Equilibrium Trade in Automobiles*

Kenneth Gillingham †
Yale University

Fedor Iskhakov ‡
Australian National University

Anders Munk-Nielsen §
University of Copenhagen

John Rust ¶
Georgetown University

Bertel Schjerning ‖
University of Copenhagen

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Abstract

We introduce a computationally tractable dynamic equilibrium model of automobile markets with heterogeneous consumers who choose to keep their car or trade for a different make, model and age. We focus on stationary flow equilibria, where outflows of cars due to accidents and endogenous scrappage equal inflows of new cars. We introduce a fast robust algorithm for computing equilibria and use it to estimate a model with eight household types and four car types using nearly 39 million observations on car ownership transitions from Denmark. The estimated model fits the data well and counterfactual simulations show that Denmark is over the top of the Laffer curve: it could *raise* total tax revenue by reducing the new car registration tax rate. We show that reducing this tax rate while raising the tax rate on fuel increases aggregate welfare, tax revenues, and car ownership, while reducing car ages, driving, and CO_2 emissions.

Keywords: secondary markets, trade, consumer heterogeneity, transactions costs, dynamic programming, extreme value distribution, dynamic discrete choice, multinomial logit model, stationary equilibrium, Markov chains, invariant distributions, doubly nested fixed point estimator (DNFXP)

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†Yale University, School of the Environment, Department of Economics, School of Management, `kenneth.gillingham@yale.edu`

‡Australian National University, Research School of Economics, `fedor.iskhakov@anu.edu.au`

§University of Copenhagen, Department of Economics and Centre for Computational Economics (CCE), `amn@econ.ku.dk`

¶**Corresponding Author:** Georgetown University, Department of Economics, Georgetown University, `jr1393@georgetown.edu`

‖University of Copenhagen, Department of Economics and Centre for Computational Economics (CCE), `bertel.schjerning@econ.ku.dk`

1 Introduction

Modeling the automobile market is particularly challenging due to the trading and substitution possibilities that exist due to the presence of a secondary market where used cars are traded. Not only are there dozens or even hundreds of different makes and models of new cars to choose from in the primary market, consumers have a huge array of used car options as well. They also decide whether to scrap or sell their car, and can respond to an increase in new car prices by switching to the outside good (i.e. not having a car), or holding their existing used car longer. Endogenous scrappage of cars is also of interest for safety and environmental reasons, since it is well documented that used cars become less safe and and pollute more as they age.[1]

We develop a dynamic model of trading in new and used cars that demonstrates how secondary markets lead to significant gains from trade via clear patterns of specialization in the holdings of cars by heterogeneous consumers. The secondary market facilitates a "hand-me-down-chain" for cars where rich consumers buy brand new cars and hold them a few years before selling to other slightly less rich consumers, who hold the car for a few more years before being traded to an even poorer consumer who may hold the car until it is involved in an accident or voluntarily scrapped.

The secondary market creates substitution possibilities that can limit market power and affect pricing decisions by new car producers in the primary market. High government taxation of new cars can also cause consumers to hold their used cars longer, as well as to substitute to the "outside good", i.e. not to own a car. Government sales taxes and regulations on emissions and safety can also interfere with the operation of secondary markets and reduce trade and consumer welfare. Beyond some point, sufficiently high taxation and overly onerous safety/emissions regulations can serve to kill off the secondary market and push consumers into the outside good. Thus there is a "Laffer curve" and the possibility of *increasing* total tax revenues by *decreasing* tax rates.

We use our model to analyze the fiscal and welfare effect of the new car registration tax in Denmark, one of the highest in the world, which in the sample period amounted to 180% of the new car price on top of a 25% value added tax (VAT). The Danish government is highly reliant on this tax, which accounts for approximately 4% of *all* tax revenues

[1] See, for example, Borken-Kleefeld and Chen (2015) and NHTSA (2013). The evidence is less clear on whether a car's fuel efficiency (measured as kilometers per liter) declines with age.

or about 2% of Danish GDP. Our model predicts that the Danish tax rate is "over the top of the Laffer curve" and that Denmark could raise more tax revenue by reducing the registration tax rate. We show that reducing new car taxes and raising fuel taxes improves aggregate welfare and significantly increases tax revenue and car ownership, while reducing average car ages and per household driving and aggregate CO_2 emissions. However the new policy is not a Pareto improvement absent offsetting transfers: the change in taxes reduces the welfare of those with long commute distances at the expense of those with shorter ones.

Our results are made possible by the fact that we can rapidly compute equilibria of the model using a fast and robust Newton-based solution algorithm that can be nested within a maximum likelihood estimation algorithm. Using Danish register data, which records the car ownership and trading decisions of all Danish citizens, we empirically estimate a version of our model with 8 types of households and 4 types of cars and show it provides a good approximation to car holdings and trading in Denmark. Our model provides a simple explanation for a striking zig-zag pattern in scrappage rates of older cars, whereby cars of even ages are scrapped with significantly higher probability compared to cars whose ages are odd numbers. We show that this is consistent with the rigorous bi-annual safety inspections in Denmark. Our estimation results reveal that these inspections have high perceived "hassle costs" so that once cars are sufficiently old, most Danes prefer to scrap their vehicles rather than incur the time and expense to repair their vehicles to pass the mandatory inspection.

The primary contribution of this paper is to advance the state of the art for computing equilibria in the primary and secondary markets for automobiles and other durable goods. We introduce a computationally tractable dynamic equilibrium model where new and used vehicles of multiple types (e.g. makes and models) are traded by heterogeneous consumers. Prices of used cars equate supply and demand for all car types and traded vintages. The ages at which cars are scrapped are also determined endogenously as part of the equilibrium. The model allows for transactions costs, taxes, and flexible specifications of car characteristics, consumer preferences, and heterogeneity. Our framework can be used to address a wide range of research and policy questions relating to the primary and secondary markets for vehicles. We also show how to incorporate a utility-based model of driving into the model, which is crucial for analyzing environmental policies.

We derive market demand from micro aggregation of an individual-level dynamic discrete choice model of ownership and trade of automobiles. Our specification of consumer heterogeneity includes additive idiosyncratic generalized extreme value preference shocks that can be interpreted as unobserved costs of maintaining an existing car, consumer-specific variations in search/transactions costs, and idiosyncratic variations in transaction prices and other costs involved in trading cars that constitute an important source of gains from trade that explain the existence of secondary markets. By varying the scale of these additive extreme value preference shocks, we show how reductions in consumer heterogeneity reduce gains from trade and ultimately kill off secondary markets when trade frictions are sufficiently large.

The generalized extreme value specification results in logit or nested logit conditional choice probabilities for the decisions to keep or trade different types and ages of vehicles. We show how additional persistent consumer heterogeneity can be added, giving us the flexibility to match rich patterns of trading, including consumers who choose not to own cars (i.e. "outside good") and brand loyalty and brand switching behaviors. We show that the choice probabilities, and thus aggregate demand, are smooth functions of car prices which allows us to use fast derivative-based methods such as Newton's method to solve for consumers' dynamic trading strategies and equilibrium prices.

We formulate our model and define equilibrium in an infinite horizon stationary environment. We use the machinery of Markov processes to describe trading behavior and characterize the vehicle holdings of different types of consumers as invariant distributions to certain Markov chains. These Markov chains reflect the trading of vehicles, their aging and the impact of stochastic accidents that result in premature scrappage of some vehicles. Our stationary equilibrium concept results in a very compact and elegant description of equilibrium that can be extended to non-stationary environments with macroeconomic shocks and overlapping generations of consumers with finite lifespans.

Section 2 reviews the large theoretical and empirical literature on modeling auto markets and other durable goods on which we build. Section 3 introduces the basic model with multiple car brands and idiosyncratic consumer heterogeneity, and Section 4 adds persistent consumer heterogeneity that increases trade of cars between different consumer types. Section 5 describes how the model parameters can be structurally estimated by maximum likelihood using a doubly nested fixed point algorithm (DNFXP) that recomputes equilibrium prices, holdings and consumer trading strategies each time the likelihood function is evaluated. We also establish the identification of the structural parameters. In Section 6 we estimate the model using Danish register data and analyze the welfare and environmental impacts of changes in Danish car tax policies. Section 7 concludes with a discussion of various directions in which the model can be extended.

2 Previous work on modeling automobile markets

A starting point of any discussion of the literature on equilibrium models of automobile markets is the well-known BLP model (Berry, Levinsohn and Pakes, 1995). This influential work focuses on the primary market for new vehicles, but ignores the presence of the secondary market and the substitution possibilities it offers consumers. Rust (1985b) and Esteban and Shum (2007) were the first to tackle the challenging problem of solving for a full equilibrium in both the primary and secondary markets for automobiles. Rust studied the simultaneous determination of price and durability by a monopolist new vehicle producer, while Esteban and Shum studied oligopolistic pricing of competing new vehicle producers. To make progress, both of these studies assumed stationarity and zero transaction costs, which implies that consumers trade each period for their most preferred vehicle in the entire market.[2]

We build on a substantial literature focused on modeling equilibrium in secondary markets for automobiles, taking the price of new vehicles as given. The earliest work that we are aware of in this literature is a series of papers by Manski (1980), Manski and Sherman (1980), and Manski (1983). These papers introduced theoretical models of equilibrium in secondary markets for cars that could be numerically solved for prices and quantities and used for policy forecasting of a wide range of policies of interest.

The next important early contribution was Berkovec (1985), who microeconometrically estimated and numerically solved a large-scale equilibrium model of the new and used vehicle markets using a nested logit model. He defined "expected excess demand" by summing estimated discrete choice probabilities for cars of each age and class, net of scrappage.[3] Berkovec computed equilibrium prices using Newton's method to find a zero

[2]Esteban and Shum (2007) also assume quality ladder preferences, which further simplifies the choice problem.

[3]Berkovec used a probabilistic model of vehicle scrappage from Manski and Goldin (1983), where the probability a vehicle is scrapped is a decreasing function of the difference between the second-hand price of the vehicle (net of any repair costs) and an exogenously specified scrap value for the vehicle. This implies that, except for

to a system of 131 nonlinear equations representing the excess demand for the vehicles in his model.[4]

The contributions of Manski, Sherman, and Berkovec were extremely advanced given the computing power at the time, and in many respects represent the closest point of departure for our own work. However, their work was based on short run, *static* equilibrium holding models of the market. Implicit in the static discrete choice formulation is the assumption that consumers only keep their vehicle for a single period, so that at the end of each period consumers trade their current vehicle for their most preferred vehicle. Rust (1985a) formulated the first *dynamic* equilibrium model of automobile trading.[5] He assumed the state of a vehicle is captured by its odometer reading x_t, which evolves according to an exogenous Markov process representing variable usage of cars with transition probability $\Phi(x_{t+1}|x_t)$ that reflects stochastic usage and deterioration of vehicles.[6]

When there are no transaction costs and the economy is in a stationary equilibrium (i.e. no macroeconomic shocks or other time-varying factors altering the market), the optimal trading strategy involves trading every period for the most preferred age/condition of vehicle $x^*(\tau)$, where τ indexes potentially heterogeneous preferences over "newness" of vehicles. However, the assumption of zero transactions costs is unrealistic, and so is the excessive trading behavior it implies. When there are transactions costs (which are distinct from *trading costs*, i.e. the difference between the price of a car x a consumer wishes to buy, $P(x)$, less the price $P(x')$ of the car x' that the consumer wishes to sell), the optimal trading strategy involves less frequent trading and consumers will generally keep cars for multiple periods. The optimal strategy then takes the form of an "(S, s) rule" reminiscent of optimal inventory theory: trade is characterized by two thresholds $(\underline{x}^*(\tau), \overline{x}^*(\tau))$, where $\underline{x}^*(\tau) < \overline{x}^*(\tau)$ and $\underline{x}^*(\tau)$ is the state of the optimal

random accidents, there is very little chance that new vehicles are scrapped, but the probability a used vehicle is scrapped increases monotonically with the age of the vehicle.

[4] Berkovec showed that the Jacobian matrix had special structure he called "identity outer product" that enabled him to invert the Jacobian via inverting a smaller 48×48 matrix and doing some additional matrix vector multiplications.

[5] Other dynamic models of vehicle choice appeared around this time such as Mannering and Winston (1985) but their analysis focused on dynamics of utilization, but did not consider dynamics of car trading or equilibrium. In subsequent work Winston and Yan (2021) develop empirically estimable model of dynamics of utilization and trading of cars, but in a partial-equilibrium framework.

[6] Since x_t fully captures the state of a car and is observable by both parties in a transaction, Rust's analysis avoided "lemons problem" information asymmetries of the type analyzed in the seminal work of Akerlof (1970) that can potentially kill off the secondary markets for cars.

replacement vehicle whenever the consumer trades in for a new one. $\overline{x}^*(\tau)$ is the *replacement threshold* or odometer threshold where it is optimal to trade the current car in condition x for a replacement car in condition $\underline{x}^*(\tau)$. When transactions costs are zero, then $\overline{x}^*(\tau) = \underline{x}^*(\tau) = x^*(\tau)$ and it is optimal to trade for the optimal car $x^*(\tau)$ every period. However, in a homogeneous agent economy, the slightest transaction costs will completely kill off the secondary market, driving all consumers into an autarkic "buy and hold" equilibrium where all consumers buy brand new vehicles whenever they trade (i.e. $\underline{x}^*(\tau) = 0$) and hold them until it is optimal to scrap their current car when the odometer exceeds an optimal replacement threshold $\overline{x}^*(\tau)$.

There are potential gains from trade in a heterogeneous agent economy that enable the existence of secondary market and a wide range of car trading strategies. However establishing the existence of a stationary equilibrium in such an economy in the presence of transactions costs is challenging. Consider a consumer of type τ who desires to buy a vehicle with $\underline{x}^*(\tau) > 0$. When there are transactions costs there is no guarantee that some other consumer τ' is willing to sell their vehicle at $\underline{x}^*(\tau)$. Using advanced methods from functional analysis (e.g. the Fan-Glicksburg fixed point theorem), Konishi and Sandfort (2002) established the existence of a stationary equilibrium in the presence of transactions costs under certain conditions. Their proof shows that it is possible for the equilibrium price function $P(x)$ to adjust to prevent such coordination failures. However to our knowledge, there has been no work actually calculating equilibria with transactions costs in this infinite-dimensional setting.

Stolyarov (2002) advanced the literature by assuming that the state of a vehicle can be summarized by its age a, which can take only a finite number of values, $a = 0, 1, 2, \ldots, \bar{a}$, where $\bar{a}$ is age when cars are scrapped. Stolyarov introduced a continuous uni-dimensional parameterization of consumer heterogeneity with quasi-linear preferences, and computed equilibria in the presence of stochastic transactions costs using a fixed point formulation of the problem. Gavazza, Lizzeri and Roketskiy (2014) extended Stolyarov's approach by allowing households to own up to two vehicles using a two-dimensional specification of consumer heterogeneity. They find that transaction costs have a large effect on equilibrium trade.[7]

[7]There is a close connection between models of automobile trading that incorporate transactions costs and models that emphasize information asymmetries, such as Akerlof (1970). House and Leahy (2004) show how adjustment costs of the (S, s) variety discussed above "arise endogenously from adverse selection in the secondary

Our model can be thought of as combining Stolyarov (2002) with the earlier work by Manski and Berkovec by using a multi-dimensional extreme value specification to capture idiosyncratic consumer heterogeneity. We use a hierarchical specification of heterogeneity that includes both time-varying idiosyncratic preference shocks (i.e. the extreme value error terms in the model) as well as flexible specifications for persistent heterogeneity and fixed consumer types τ. The extreme value distribution allows for continuous formulas for choice probabilities even in the case where there is no other time-invariant heterogeneity, and this continuity permits us to demonstrate the existence of equilibrium via the Brouwer fixed point theorem. More importantly, we show that the excess demand function for used cars in our model, $ED(P)$, is a continuously differentiable function of P that enables us to rapidly and accurately calculate equilibrium prices by solving the system of nonlinear equations $ED(P) = 0$ by Newton's method. This makes our approach very attractive for use in empirical work and policy modeling.

3 Equilibrium with Idiosyncratic Consumer Heterogeneity

In this section we introduce a dynamic model of equilibrium trade in the automobile market. We use the concept of *stationary flow equilibrium* in the market of stochastically deteriorating durable goods from Rust (1985a) but adapt it for the discrete goods trade in presence of flexible transactions costs. We start by considering equilibrium with J different makes/models of cars and a unit mass of consumers whose preferences for cars as well as the outside option, are *idiosyncratically heterogeneous.* We adopt a generalized extreme value (GEV) specification of consumer heterogeneity that results in a nested logit specification for choice probabilities similar to Berkovec (1985). In subsequent section we extend the framework to persistent heterogeneity in consumer preferences.

market." (p. 582). For example, there are "lemons laws" in many countries that require sellers to compensate buyers for defects or problems in a car that were not disclosed and negotiated on at the time of sale. Dealers typically perform inspections and repair cars before selling, and often provide a limited term warranty, all of which mitigate the informational asymmetries and result in transactions costs that are often borne by the dealer. As a result, it is not clear that informational asymmetries seriously inhibit trade in used vehicles, but they would be expected to show up in transaction costs. Hendel and Lizzeri (1999) study equilibria in auto markets with and without asymmetric information and find that adverse selection does not necessarily kill off the secondary market. They find it difficult to empirically distinguish between predictions of models with asymmetric information and those with transaction costs, and argue that, for Fords and Hondas at least, the evidence does not support adverse selection as the primary reason for steeper price declines of Fords as the vehicles age. In light of this, we use transaction costs to capture various trade frictions in auto markets including informational ones.

3.1 Key assumptions and restrictions

We consider a stationary equilibrium in an infinite horizon economy where cars are initially sold as new in the primary market and then traded in used car markets called "secondary markets". Consumers make purchase, replacement, trading and scrapping decisions to maximize expected discounted utility with a common discount factor $\beta \in (0, 1)$. We focus on a stationary environment and do not allow for any "macro shocks" that could lead to time-varying fuel prices or prices of new cars.[8]

Our concept of equilibrium results in endogenous determination of a vector of equilibrium prices P with typical element P_{ja}, where $j \in \{1, \dots, J\}$ indexes makes/models, and $a \in \{1, \dots, \bar{a} - 1\}$ indexes the ages of the traded cars. When the cars reach the upper bound $\bar{a}$, they are no longer safe to drive and are not allowed to be kept or traded and must be scrapped.[9] We treat the model as a "small open economy" where new car prices are determined in the world market with an infinitely elastic supply of new cars at prices $\overline{P}_j$.[10] We assume there is an infinitely elastic demand for cars *at any age* including $\bar{a}$ for their scrap value $\underline{P}_j$ which normally results in $P_{ja} \geq \underline{P}_j$, $\forall j, a$, provided that the level of transactions costs is not too high.

In our framework all persistent differences between the cars are captured by the make/model $j \in \{1, \dots, J\}$, and all time-varying characteristics of cars are reflected by the car age $a \in \{1, \dots, \bar{a}\}$. The unit mass of cars in the economy is distributed among $J\bar{a}$ types given by the combination of car make/model and age (j, a). Clearly used cars of the same age and type have idiosyncratic features, such as odometer reading which we ignore, making it inconsistent with a single common price P_{ja} for all used cars of age a and make/model j. This is partially accounted for in our framework but the stochastic GEV shocks which not only reflect idiosyncratic heterogeneity in consumer preferences, but also idiosyncratic features of different used cars of the same age and type. Thus, we can interpret P_{ja} as the average price of a car of type j and age a, and components of the idiosyncratic shocks reflect customer and car-specific deviations in these prices from

[8] Although it is possible to extend our framework to allow for macro shocks, this fundamentally changes the definition of the equilibrium. We defer this extension to the future work due to the vastly greater computational challenges that it presents, as noted in the work of Krusell and Smith (1998) and Cao (2016).

[9] The same upper bound is assumed to hold for the cars of all makes/models without loss of generality to simplify exposition. It is straightforward to allow upper bound to be j-specific with more complicated notation.

[10] However, our framework can be used for modeling competition in the primary market for new cars, where $\overline{P}_j$ can be set taking into account the substitution effects not only between different types of *new cars* but also between *new and used* cars.

the market average prices that are determined endogenously in equilibrium.

We assume that consumers' preferences are characterized by a common quasi-linear utility function

$$U(\cdot) = u(j,a) - \mu\big[\text{operating costs} + \text{trade and transaction costs}\big],$$

where the first term captures the utility of owning and using a car and the second term accounts for the monetary costs of ownership and trade. We assume that the marginal utility of an additional car is sufficiently small such that no consumer would want to own more than a single car.[11] The parameter $\mu > 0$ is a simple way to capture income/wealth effects in the model. High values of μ can be interpreted as "being poor" because the cost of buying a new car will involve a high opportunity cost in terms of forgone consumption of other goods. We expect the function $u(j,a)$ would be non-increasing in a for all j, and will later show how it captures the utility of driving (utilization) as well as the expected non-monetary cost of maintaining a car of age a.

Trade costs consist of the difference in prices of traded cars, with the addition of transaction costs. Let (i,a) be the make/model and age of the existing car, and (j,d) denote the car the household purchases. Transactions costs are given by an function $T(i,a,j,d)$ which depends on both the traded cars and on the whole set of prices $\{\overline{P}_j, \underline{P}_j\} \cup \{P_{ja}\}$, $a \in \{1,\dots,\bar{a}-1\}$, $j \in \{1,\dots,J\}$.[12] We assume that the transactions costs are borne by both buyers and sellers. Even though it would be possible to work with general non-separable specifications for transactions costs $T(i,a,j,d)$, for simplicity we assume that these costs are additively separable into two components denoted $T_b(j,d)$ and $T_s(i,a)$. The first component is associated with searching for and buying another car and the second with undertaking repairs and improvements to make the car of age a that the consumer is trying to sell acceptable to potential buyers. We do not make further restrictions on the function form of the transaction costs, so we can allow for both fixed and proportional costs, i.e. sales taxes and registration fees.

The total trade and transaction costs associated with selling car (i,a) to buy another

[11] This is a reasonable assumption for a country like Denmark, where most households only own a single car. It also greatly simplifies notation and the presentation of our model. The assumption could be relaxed to extend the model to countries like the United States, where most households own multiple cars.

[12] All quantities in the consumer choice problem depend on these prices, but for clarity we do not write this explicitly until Section 3.4.

car (j, d) are given by $P_{jd} - P_{ia} + T_b(j, d) + T_s(i, a)$. If a consumer chooses to "purge" their car and choose the outside option of not owning a car, the buyer-side components depending on (j, d) disappear from the expression, so the consumer only faces the seller-side transaction cost. Similarly, if a person without a car decides to purchase one, the seller-side components depending on (i, a) disappear from the trade costs. Finally, we assume that the trade-in price $\underline{P}_i$ for the cars being scrapped already includes all the costs associated with de-registering and transporting the clunker to the scrap yard. That is we normalize the seller-side transaction cost of scrapping to zero. Similarly, we assume that the search cost is negligible when buying a new car, and therefore normalize the buyer-side transaction cost for the new car buyers to zero.

3.2 Consumer states and choices

The *state* of a consumer in any period t is given by the vector (i, a, ϵ) where $i \in \{ø, 1, \ldots, J\}$ denotes the make/model of car the consumer owns at the start of the period, and $a \in \{ø, 1, \ldots, \bar{a}\}$ denotes its age. We use the special symbol ø to denote the state of not owning a car. The random component of the state vector ϵ incorporates the (idiosyncratic) heterogeneity in cars and consumers.

We assume that at the start of each period a consumer who owns an existing car (i, a) can choose whether to keep it, trade it for another car of make/model (j, d), or choose the outside option of not having a car at all. We assume trade occurs instantaneously at the start of each period, and thus the cars (j, d) that households hold after trading are utilized until the end of the period. Cars deterministically age from d to $d + 1$, but may be involved in total loss accidents which we model by stochastic transition to the terminal age $\bar{a}$ with probability $\alpha(j, d) \in [0, 1)$. The realized state of the car constitutes the car state at the start of period $t + 1$. Then instantaneous trading occurs and the process repeats this way for the infinite future.

The cars which reach the terminal age $\bar{a}$ by either natural aging or as a result of an accident are *exogenously* scrapped and removed from the market during the trading stage. In addition, unless the existing car (i, a) is kept, it can be *endogenously scrapped* instead of being sold on the secondary market. It would seem that all consumers would prefer to sell their existing used car in the market rather than scrap it, however there are transactions costs that a seller must incur, and the net value that a consumer might receive from selling a sufficiently old used car may be lower than the value from simply scrapping it. Our model allows consumers to choose whether to sell or scrap their existing car depending on which option they prefer, which can also include unobserved idiosyncratic inspection/repair costs that are incorporated in the GEV shocks we describe below.

Let $C(i,a)$ denote the choice set for the consumer who enters the period with the existing model i car that is a years old. If the consumer has no car ($a = \o$) they can choose to remain in the no-car state ($d = \o$), buy a new car ($d = 0$) of any type $j \in \{1,\dots,J\}$, or one of the vintages available for sale in the secondary market. If the consumer already owns a car model i of age $a < \bar{a}$, they have an additional option of keeping it which we denote $d = \kappa$. However, once a car reaches the terminal age $\bar{a}$ it is no longer possible to keep it according to our assumption of exogenous scrappage.[13] Every time an existing car is traded, the consumer chooses to either sell it on the secondary market which we denote by $s = 0_s$, or to take it to the scrap yard we denote by $s = 1_s$. The set of feasible choices for a consumer in a car state (i,a) is thereby summarized as follows[14]:

$$\begin{aligned} C(\o) = C(i,\bar{a}) &= \{\o\} \ \cup \ \{1,\dots,J\} \times \{0,1,\dots,\bar{a}-1\} \ , \forall j \qquad (1) \\ C(i,a) &= \{(\o,1_s),(\o,0_s),\kappa\} \ \cup \ \{1,\dots,J\} \times \{0,1,\dots,\bar{a}-1\} \times \{1_s,0_s\} \ , \forall j, a < \bar{a}, \end{aligned}$$

Since cars can only be scrapped when they reach the upper bound on age $\bar{a}$, they cannot traded, and therefore the oldest car that be purchased in the secondary market is $\bar{a}-1$ periods old.

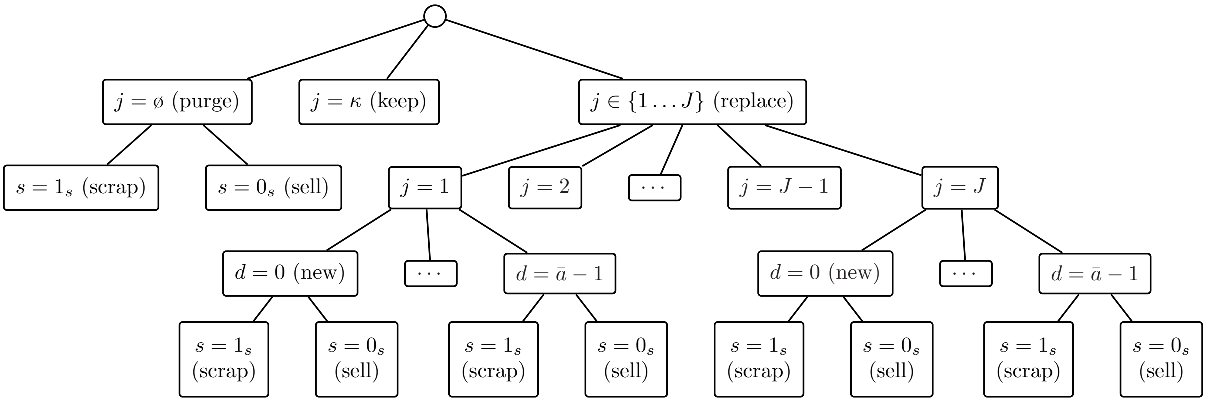
The random component ϵ is a vector whose dimension equals to the number of elements in the choice set. We assume that ϵ has a multivariate GEV distribution which allows for flexible dependence between its elements, but that the vectors are drawn independently between time periods and individuals, capturing the idiosyncratic heterogeneity between them. The elements of the vector ϵ in turn capture the differences between the discrete choices available to each individual, such as maintenance expenditures, search costs, and variability in the prices of traded cars that reflect their idiosyncratic features.[15]

[13] The difference between the decision $d = \kappa$ to keep the current car of age a and the decision $d = a$ to trade for another car of the same age and model $j = i$ is in the incurred transaction costs.

[14] To simplify notation here and throughout the paper we use single $\o$ symbol for the no car state, i.e. $C(\o)$ instead of $C(\o,\o)$.

[15] The components of ϵ should be interpreted in an *ex ante* sense, as the idiosyncratic utility/disutility the consumer can expect from undertaking a search for a used car of a given type. We do not explicitly model the

Figure 1: Example of consumer choice tree



Notes: The figure presents an example of the choice tree for a consumer who owns a car under the nested logit specification of the GEV distribution of the idiosyncratic heterogeneity term ϵ. Note three choice nests at the top level (to purge, keep or trade the existing car), two intermediate levels of nesting in the case of trading, and a additional scrappage choice of the existing car in the cases when it is traded.

The GEV distribution we apply in our framework generalizes the standard multivariate extreme value distribution and results in the choice probabilities that take the form that McFadden (1981) called the *nested multinomial logit* (NMNL) rather than standard multinomial logit (MNL). Under this specification it is possible to control for correlation patterns in different subsets of the overall choice set $C(i, a)$, and the choice itself can be represented as a sequence of choices from nested subsets (i.e. a filtration) of $C(i, a)$. However, even with this representation, all decisions are made *simultaneously* and instantaneously at the beginning of each period as described above. The patters of interdependence of the GEV distribution can be illustrated by a *choice tree,* a directed acyclical graph such as the one shown in Figure 1.

The choice tree illustrated in Figure 1 is one of the many possible ways to introduce dependence patterns into the components of the idiosyncratic shocks ϵ. This particular tree is for a consumer who owns a car, and has four levels: the top level consists of the choices to purge ($j = \o$), keep ($j = \kappa$) or replace ($j \in \{1, \ldots, J\}$) the current car. Conditional on the decision to replace, the second level is the choice of the make/model j of the replacement vehicle. The third level contains the choice of the age d of the car to buy, a new car $d = 0$. Finally, the fourth level contains the choice of whether to sell or

sequential search process in this paper, nor do we model the "microstructure" of auto dealers and other places consumers go to search for and buy used cars.

scrap the existing car if it is not kept.[16] Each level choice, apart from top choice $j = \kappa$ in our example, there is a subtree of lower level choices eventually leading to a distinct *leaf* of the tree corresponding to a particular alternative in the choice sets defined in (1).

Dependence patterns in the distribution of the elements of ϵ are determined by a set of scale parameters which can be defined individually for the alternatives immediately below the top node of each subtree of the overall choice tree. For example in Figure 1, the parameter σ controls the scale of idiosyncratic shocks at the top level choice set $\{\varnothing, \kappa, \{1, \ldots, J\}\}$ involving the decision to have no car, keep the current car, or trade for some other new or used car. The parameter σ_r controls the degree of similarity in the unobserved shocks affecting the choice of one of the J different types of cars, i.e. idiosyncratic heterogeneity in "brand effects". The parameters σ_j, $j \in \{1, \ldots, J\}$ control the scale of idiosyncratic shocks affecting the choices of different ages of cars of a given make/model. Finally the parameter σ_s controls the scale of idiosyncratic shocks reflecting unobserved components of transaction, inspection and repair costs involved in selling the current car versus scrapping it. McFadden (1981) showed that in order for the GEV distribution to be a valid multivariate probability distribution the similarity parameters must form a non-increasing sequence along any particular branch. In our example this implies that $\sigma \geq \sigma_r \geq \sigma_j \geq \sigma_s$ for all $j \in \{1, \ldots, J\}$. As any of the similarity parameters approaches zero, the choice in the corresponding choice subset becomes deterministic, as do the choices in the subtrees below. When $\sigma = \sigma_r = \sigma_j = \sigma_s$ the choice tree collapses to a one level tree with all alternatives in the chocie set $C(i, a)$ on the same level. This is the MNL case with the implied *Independence from Irrelevant Alternatives* (IIA) property: i.e. there is no dependence among the components of ϵ.

3.3 Consumer dynamic choice model

The optimal trading/holding strategy for cars is given by the solution of their infinite horizon expected utility maximization problem, which constitutes a discrete choice dynamic programming problem (Rust, 1987, 1994). Let $V(i, a, \epsilon)$ be the value function for a consumer in state (i, a, ϵ), $i \in \{\varnothing, 1, \ldots, J\}$, $a \in \{\varnothing, 1, \ldots, \bar{a}\}$. For a consumer who does

[16] The fourth level scrappage choice appears on level two for the top level decision to purge without loss of generality.

not own a car it is given by:

$$V(\o, \epsilon) = \max \left[v(\o, \o) + \epsilon(\o); \max_{\substack{j \in \{1,\ldots,J\}, \\ d \in \{0,1,\ldots,\bar{a}-1\}}} \big[v(\o, j, d) + \epsilon(j, d) \big] \right], \tag{2}$$

where the *choice specific value functions* of remaining in the no car state and of leaving the no car state to buy a car of type j age d respectively are given by:

$$\begin{aligned} v(\o, \o) &= u(\o) + \beta EV(\o), \\ v(\o, j, d) &= u(j, d) - \mu \big[P_{jd} + T_b(j, d) \big] \\ &\quad + \beta \big(1 - \alpha(j, d)\big) EV(j, d+1) + \beta \alpha(j, d) EV(j, \bar{a}). \end{aligned} \tag{3}$$

The *expected value functions* $EV(\o)$ and $EV(j, a)$ provide the conditional expected values of starting the next period respectively without a car, and with car of type i and age a, and are given by:

$$EV(\o) = \int_\epsilon V(\o, \epsilon) f(\epsilon | \o) d\epsilon, \quad EV(i, a) = \int_\epsilon V(i, a, \epsilon) f(\epsilon | i, a) d\epsilon, \tag{4}$$

where $f(\epsilon|\cdot)$ is the corresponding probability density function of a GEV distribution for the idiosyncratic shocks ϵ. Implicit in these formulas is the assumption that idiosyncratic shocks affecting the consumer's choice are independent of their past realizations. This implies that the $EV(\cdot)$ functions only depend on the car that has been driven and aged during the current period and constitutes the car state in the beginning of the next period. Under our assumption of GEV distribution of idiosyncratic shocks ϵ, the integrals in (4) can be expressed in closed-form. The formulas depend on the assumed nesting structure of the choices. Later in this section, we provide the analytic formulas for $EV(\cdot)$ corresponding to the nesting structure in Figure 1.

The value function for a consumer who starts the period with a car of terminal age $\bar{a}$ is similarly given by:

$$V(i, \bar{a}, \epsilon) = \max \left[v(i, \bar{a}, \o) + \epsilon(\o); \max_{\substack{j \in \{1,\ldots,J\}, \\ d \in \{0,1,\ldots,\bar{a}-1\}}} \big[v(i, \bar{a}, j, d) + \epsilon(j, d) \big] \right], \tag{5}$$

where again the first component corresponds to the decision to choose the outside option of not owning a car, and the second to the purchase of a new car of type j and age d. Because at terminal age $\bar{a}$ the car is scrapped exogenously, the consumer does not have the option of keeping their car, and also does not have an additional choice of endogenous scrappage of the existing car $(i, \bar{a})$. This is why the the Bellman equation (5) looks very similar to that of the consumers who do not own a car (2). The difference, however, is in the net scrap value $\underline{P}_i$ the consumer receives from the scrap yard net of towing and other costs of scrapping. Therefore the relevant choice specific value functions are given by:

$$\begin{aligned} v(i,\bar{a},\emptyset) &= u(\emptyset) + \mu \underline{P}_i + \beta EV(\emptyset), \\ v(i,\bar{a},j,d) &= u(j,d) - \mu\big[P_{jd} - \underline{P}_i + T_b(j,d)\big] \\ &\qquad + \beta\big(1-\alpha(j,d)\big)EV(j,d+1) + \beta\alpha(j,d)EV(j,\bar{a}). \end{aligned} \tag{6}$$

Finally, the value function for a consumer who starts the period with a car of type i and age a is given by:

$$V(i,a,\epsilon) = \max \begin{bmatrix} v(i,a,\kappa) + \epsilon(\kappa); \\ \max_{s\in\{1_s,0_s\}} \big[v(i,a,\emptyset,s) + \epsilon(\emptyset,s)\big]; \\ \max_{\substack{j\in\{1,\dots,J\},\\ d\in\{0,1,\dots,\bar{a}-1\},\\ s\in\{1_s,0_s\}}} \big[v(i,a,j,d,s) + \epsilon(j,d,s)\big] \end{bmatrix}. \tag{7}$$

In this case the endogenous scrappage choice $s \in \{1_s, 0_s\}$ has to be accounted for, so the complete set of choice specific value functions which correspond to all the alternatives in the choice set $C(i, a)$ defined in (1), is given by:

$$\begin{aligned} v(i,a,\emptyset,1_s) &= u(\emptyset) + \mu \underline{P}_i + \beta EV(\emptyset), \\ v(i,a,\emptyset,0_s) &= u(\emptyset) + \mu[P_{ia} - T_s(i,a)] + \beta EV(\emptyset), \\ v(i,a,\kappa) &= u(i,a) + \beta\big(1-\alpha(i,a)\big)EV(i,a+1) + \beta\alpha(i,a)EV(i,\bar{a}), \\ v(i,a,j,d,1_s) &= u(j,d) - \mu[P_{jd} - \underline{P}_i + T_b(j,d)] \\ &\qquad + \beta\big(1-\alpha(j,d)\big)EV(j,d+1) + \beta\alpha(j,d)EV(j,\bar{a}), \\ v(i,a,j,d,0_s) &= u(j,d) - \mu[P_{jd} - P_{ia} + T_s(i,a) + T_b(j,d)] \\ &\qquad + \beta\big(1-\alpha(j,d)\big)EV(j,d+1) + \beta\alpha(j,d)EV(j,\bar{a}). \end{aligned} \tag{8}$$

Here $v(i, a, \emptyset, 0_s)$ is the value of selling the car on the market and not replacing it,

$v(i, a, \varnothing, 1_s)$ is the value of scrapping the car and not replacing it — in both of these cases the customer has no car to drive and ends up in the no car state in the next period.[17] The value of keeping the existing car is given by $v(i, a, \kappa)$, and the values of trading the existing car (i, a) to a replacement car (j, d) is denoted $v(i, a, j, d, 0_s)$ and $v(i, a, j, d, 1_s)$, respectively, depending on whether the existing car is sold or scrapped.

Combined, the value functions defined in (2), (5) and (7) cover the whole state space of the problem (i, a, ϵ), $i \in \{\varnothing, 1, \ldots, J\}$, $a \in \{\varnothing, 1, \ldots, \bar{a}\}$. With their corresponding choice specific values, and the general formula for the expected value function (4), we can define the Bellman equation for the consumer choice problem as a mapping of the space of value functions $V(i, a, \epsilon)$ to itself, and standard contraction mapping arguments guarantee that V is the unique fixed point to the "Bellman operator". However, the problem can be solved in a computationally much easier fashion in terms of the "projection" $EV(i, a)$ which as we noted above is a much lower dimensional object since it does not depend on the continuously distributed idiosyncratic state variables ϵ.[18] EV is just a finite dimensional vector, $EV \in \mathbb{R}^{J\bar{a}+1}$, whose elements are the expected values of starting the next period in car state (i, a), $i \in \{1, \ldots, J\}$, $a \in \{1, \ldots, \bar{a}\}$, or with no car.[19] Applying equations (4) for each element of vector EV, plugging in the expressions (2), (5), (7) and their corresponding choice specific value functions, we derive the system of $J\bar{a} + 1$ nonlinear equations

$$EV = \Gamma(EV), \tag{9}$$

whose solution enables us to reconstruct V and characterize optimal trading behavior. Here Γ is the *smoothed Bellman operator* that constitutes a contraction mapping and hence has a unique fixed point. Further, Γ is a smooth mapping from $\mathbb{R}^{J\bar{a}+1}$ to $\mathbb{R}^{J\bar{a}+1}$, which enables us to use Newton's method in combination with the method of successive approximations to rapidly compute this unique finite-dimensional fixed point and thus solve the consumer choice problem for any set of car prices.[20]

[17] Recall that by our assumption the transaction cost of selling to the scrap yard is normalized to zero, i.e. included into $\underline{P}_i$.

[18] Another though inferior possibility is to formulate and solve the Bellman equation in the space of choice specific value functions, which depend on both state and choice variables, and thus constitute even higher dimensional object than the value functions themselves. All three ways to set up the fixed point problem are equivalent in the sense that they lead to the same solution, and the corresponding functional mappings are contractions (Ma and Stachurski, 2021).

[19] Recall that due to our timing assumption it is not possible to start the period with a new car ($a = 0$): all new cars purchased during the trading stage become one year old cars by the start of the next period.

[20] See Lemma L2 for details and proof.

Under the GEV distributional assumption, the integrals in the equation (4) have closed form expressions, further contributing to the computational tractability of the problem. The analytic form of these depends on the assumed nesting of choices in the decision process, as in the example illustrated in Figure 1, and the implied dependency structure of the elements of ϵ. Generally, the choice specific values $v(i, a, \cdot)$ which correspond to the bottom layer of the nodes in the tree, are combined together following the tree structure with the help of McFadden's (1981) *log-sum* (smoothed max) function $\mathcal{I}(\cdot)$, defined as:

$$\mathcal{I}(\lambda, \chi_1, \ldots, \chi_n) = \lambda \log \left(\exp \tfrac{\chi_1}{\lambda} + \cdots + \exp \tfrac{\chi_n}{\lambda}\right), \tag{10}$$

where λ takes the value of the scale parameter in each particular grouping of alternatives (nest). Using the example choice tree in Figure 1, we have:

$$EV(i,a) = \mathcal{I}\Big(\sigma, \underbrace{\mathcal{I}\big(\sigma_s, v(i,a,\emptyset,1_s), v(i,a,\emptyset,0_s)\big)}_{\text{purge}}, \underbrace{v(i,a,\kappa)}_{\text{keep}}, \underbrace{\mathcal{I}\big(\sigma_r, I_1, \ldots, I_J\big)}_{\text{replace}}\Big), \tag{11}$$

where I_j are the *inclusive values* of trading to a car make/model j, which are found by recursively applying the log-sum function (10) to the further nests as:

$$I_j = \mathcal{I}\big(\sigma_j, \underbrace{I(i,a,j,0,)}_{\text{new car}}, \ldots, I(i,a,j,\bar{a}-1)\big),$$

where $I(i,a,j,d) = \mathcal{I}\big(\sigma_s, v(i,a,j,d,1_s), v(i,a,j,d,0_s)\big)$ is the inclusive value of the nested scrappage decision corresponding to the choice of car j of age $d \in \{0, \ldots, \bar{a}-1\}$, including the new car. When scale parameters are equalized, $\sigma = \sigma_r = \sigma_j = \sigma_s$, the above nested log-sum functions collapse and the expected value involves a single log-sum formula where the choice specific values correspond to all alternatives in the choice set $C(i,a)$. The expected values $EV(\emptyset)$ and $EV(i,\bar{a})$ involve similar closed-form expressions.

Let $\Pi(j,d,s|i,a)$ be the conditional probability of choosing a feasible alternative (j,d,s) from the choice set $C(i,a)$ by a consumer in a given car state (i,a). Under the GEV assumption these choice probabilities take NMNL closed form expressions which also depend on the structure of the choice tree (McFadden, 1981). For example, under the choice structure in Figure 1 the top level probability of keeping the existing car follows directly from (11):

$$\Pi(\kappa|i,a) = \tag{12}$$
$$\frac{\exp(v(i,a,\kappa)/\sigma)}{\exp(\mathcal{I}(\sigma_s, v(i,a,\emptyset,1_s), v(i,a,\emptyset,0_s))/\sigma) + \exp(v(i,a,\kappa)/\sigma) + \exp(\mathcal{I}(\sigma_r, I_1,\dots,I_J)/\sigma)}$$

For more complicated nested choices such as the choice of scrapping/selling the existing car (i,a) and replacing it with car (j,d) following the choice tree in Figure 1 the choice probability $\Pi(j,d,s|i,a)$ can be decomposed into products of conditional probabilities as:

$$\Pi(j,d,s|i,a) = \Pi(\text{replace}|i,a)\cdot\Pi(j|\{1,\dots,J\},i,a)\cdot\Pi(d|j,i,a)\cdot\Pi(s|j,d,i,a), \tag{13}$$

where $\Pi(\text{replace}|i,a)$ denotes the probability of trading for *some* type of car, the choice probability $\Pi(j|\{1,\dots,J\},i,a)$ corresponds to choosing a particular make/model j conditional on having decided to replace the existing car (i,a), $\Pi(d|j,i,a)$ is the choice probability for a particular age d, and $\Pi(s|j,d,i,a)$ is the probability of the choice of scrapping the existing car (i,a) or selling it in the secondary market. These probabilities are given by the following expressions:

$$\Pi(\text{replace}|i,a) = \tag{14}$$
$$\frac{\exp(\mathcal{I}(\sigma_r, I_1,\dots,I_J)/\sigma)}{\exp(\mathcal{I}(\sigma_s, v(i,a,\emptyset,1_s), v(i,a,\emptyset,0_s))/\sigma) + \exp(v(i,a,\kappa)/\sigma) + \exp(\mathcal{I}(\sigma_r, I_1,\dots,I_J)/\sigma)},$$

$$\Pi(j|\{1,\dots,J\},i,a) = \frac{\exp(I_j/\sigma_r)}{\exp(I_1/\sigma_r)+\cdots+\exp(I_J/\sigma_r)}, \tag{15}$$
$$\Pi(d|j,i,a) = \frac{\exp(I(i,a,j,d)/\sigma_j)}{\exp(I(i,a,j,0)/\sigma_j)+\cdots+\exp(I(i,a,j,\bar{a}-1)/\sigma_j)}, \tag{16}$$
$$\Pi(s|j,d,i,a) = \frac{\exp(v(i,a,j,d,s)/\sigma_s)}{\exp(v(i,a,j,d,1_s)/\sigma_s)+\exp(v(i,a,j,d,1_s)/\sigma_s)}. \tag{17}$$

The remaining choice probabilities for all feasible alternatives in the choice sets (1) corresponding to all other states in the model (including having no car $i=\emptyset$ or having the car of terminal age $a=\bar{a}$), have similar closed form expressions which combine the corresponding choice-specific value functions implied by the assumed nesting structure of choices.

Due to additively separable transaction costs, and provided that the scale parameter σ_s

is the same in all nests of the choice tree where the decision is relevant, the choice between scrapping and selling the existing car is independent of the choice of the replacement car and has no implications for the future periods. This implies that the scrap/sell decision is static, so $\Pi(s|j,d,i,a) = \Pi(s|i,a)$ and we can generally factor any choice probability as $\Pi(j,d,s|i,a) = \Pi(j,d|i,a)\Pi(s|i,a)$, where we refer to $\Pi(j,d|i,a)$ as the trading choice probability, and $\Pi(s|i,a)$, $s \in \{1_s, 0_s\}$ as the endogenous scrappage probability. It is readily verified that the scrappage probability is the same for both the case of purging and replacing the current car (i,a), so for all $i \in \{1,\dots,J\}$, $a \in \{1,\dots,\bar{a}-1\}$, it is given by:

$$\Pi(1_s|i,a) = \Big(1 + \exp\big(\tfrac{\mu}{\sigma_s}[P_{ia} - T_s(i,a) - \underline{P}_i]\big)\Big)^{-1}. \tag{18}$$

In other words, car owners choose to scrap or sell their existing car based on the difference between the market price net of seller transactions cost and the scrap value of their car, conditional the marginal utility of money μ and the scale parameter σ_s.

3.4 Equilibrium with idiosyncratic consumer heterogeneity

With the consumer dynamic choice problem fully described, in this section we turn to the definition of stationary equilibrium in the secondary market for automobiles. Recall that we assume that the prices of the new cars $\overline{P}_j$, $j \in \{1,\dots,J\}$ are fixed, and that the supply of new cars is infinitely elastic. Similarly we assume that there is an infinitely elastic demand for cars for their scrap value $\underline{P}_j$, and so we also treat the scrap value of a car of each type j as fixed. We also assume that all cars have to be scrapped at the upper bound age $\bar{a}$.

The used cars of age from $a = 1$ to $a = \bar{a} - 1$ of each make/model are traded in the secondary market. Supply and demand of these $J(\bar{a}-1)$ tradable goods are balanced by $J(\bar{a}-1)$ prices which we combine into the J-block price vector

$$P = \big(P_1,\dots,P_J\big) = \Big(\big(P_{11},\dots,P_{1\bar{a}-1}\big),\dots,\big(P_{J1},\dots,P_{J\bar{a}-1}\big)\Big) \in \mathbb{R}^{J(\bar{a}-1)}. \tag{19}$$

The value functions and choice probabilities derived in the previous subsection are all implicit functions of P, though we suppressed their dependence on P so as not to overload the notation.

Let $0 \le q_{ia} \le 1$ denote the fraction of the unit mass of households in car state

(i, a), namely those who own the car of make/model i of age a at the start of the period before the trading phase. Let $0 \leq q_\emptyset \leq 1$ be the fraction of households without a car. The *ownership distribution* vector q summarizes the distribution of the unit mass of consumers in the economy over all possible car states:

$$q = \big(q_1, \ldots, q_J, q_\emptyset\big) = \Big((q_{11}, \ldots, q_{1\bar{a}}), \ldots, (q_{J1}, \ldots, q_{J\bar{a}}), q_\emptyset\Big) \in \mathbb{R}^{J\bar{a}+1}. \tag{20}$$

The ownership distribution q is a proper probability vector (its elements sum up to 1), and thus belongs to the $J\bar{a}$-dimensional unit simplex. The subvectors of the ownership distribution q_i correspond to the particular makes/models of the car, and do not represent a proper distribution unless normalized. The conditional distribution (i.e. market shares) of all cars in the economy is a vector with one less element than q, and can be constructed by using its first $J\bar{a}$ elements, multiplied with the normalization constant $1/(1 - q_\emptyset)$.

Though we have a continuum of consumers, we are studying an economy with a finite number of goods, so our concept of equilibrium involves the traditional approach of finding a vector P that equates supply and demand for all used cars in the secondary market. However with a continuum of consumers, we will define supply and demand in terms of the *fraction* of the total population of consumers who wish to sell and to buy a car of a given type and age, (j, d).[21] Let $D_{jd}(P, q)$ be the demand for make/model j cars of age d. Conditional on the ownership distribution of consumers q, for $j \in \{1, \ldots, J\}$, $d \in \{1, \ldots, \bar{a} - 1\}$, it is given by

$$D_{jd}(P, q) = \Pi(j, d|\emptyset, P)q_\emptyset + \sum_{i=1}^{J} \sum_{a=1}^{\bar{a}} \Pi(j, d|i, a, P)q_{ia}, \tag{21}$$

where we now include P as an argument of the choice probabilities $\Pi(j, d|\cdot, P)$ to emphasize their dependence on market prices.

The supply of used cars to the secondary market are those which are not kept and not scrapped. The corresponding fractions of consumers are given by the complements to the choice probability of keeping, $\Pi(\kappa|i, a, P)$ and scrapping, $\Pi(1_s|i, a, P)$. Let $S_{jd}(P, q)$ be the supply for the cars of make/model j of age d, $j \in \{1, \ldots, J\}$, $d \in \{1, \ldots, \bar{a} - 1\}$.

[21] Each consumer in the economy chooses the alternative that maximizes their payoff conditional on their independent draw of the random component ϵ; by the Law of Large Numbers, these *deterministic* individual choices aggregate to population shares that are defined in terms of q and choice probabilities.

It is given by

$$S_{jd}(P,q) = \big(1 - \Pi(\kappa|j,d,P)\big)\big(1 - \Pi(1_s|j,d,P)\big)q_{jd}. \tag{22}$$

Supply and demand of the cars of all make/models and all ages can be stacked in the same way as price vector (19) to form the J-block vectors of demand and supply, $D(P,q)$ and $S(P,q)$. We then define the vector of *excess demand* as

$$ED(P,q) = \big(D_{11}(P,q) - S_{11}(P,q), \ldots, D_{J,\bar{a}-1}(P,q) - S_{J,\bar{a}-1}(P,q)\big) \in \mathbb{R}^{J(\bar{a}-1)} \tag{23}$$

In equilibrium the prices equate supply and demand of used cars resulting in zero excess demand, so equilibrium prices are a solution to the non-linear system of $J(\bar{a}-1)$ equations given by $ED(P,q) = 0$ with $J(\bar{a}-1)$ unknown prices P.

Besides the market clearing condition which has to hold in each time period, in a stationary flow equilibrium we require the ownership distribution q to be time-invariant. The evolution of q can be broken into two stages: 1) an instantaneous trading phase in the beginning of each period, and 2) the rest of the period when car utilization takes place. After the trading phase ownership of cars changes due to trade between households. Also old cars are scrapped and new cars purchased. Then between periods t and $t+1$ the state of cars change as they either become one period older or are involved in an accident.

To describe the two phases of the evolution of the ownership distribution we rely on the tools from Markov chain theory, and describe car ownership and state transitions using two transition probability matrices Q and $\Omega(P)$ defined below. Let $\Omega(P)$ be the $(J\bar{a}+1) \times (J\bar{a}+1)$ *trade transition probability matrix* given by

$$\Omega(P) = \begin{bmatrix} \Delta_{11}(P) + \Lambda_1(P) & \Delta_{12}(P) & \ldots & \Delta_{1J}(P) & \Delta_{1\emptyset}(P) \\ \Delta_{21}(P) & \Delta_{22}(P) + \Lambda_2(P) & \ldots & \Delta_{2J}(P) & \Delta_{2\emptyset}(P) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Delta_{J1}(P) & \Delta_{J2}(P) & \ldots & \Delta_{JJ}(P) + \Lambda_J(P) & \Delta_{J\emptyset}(P) \\ \Delta_{\emptyset 1}(P) & \Delta_{\emptyset 2}(P) & \ldots & \Delta_{\emptyset J}(P) & \Pi(\emptyset|\emptyset,P) \end{bmatrix}, \tag{24}$$

where the typical $\bar{a} \times \bar{a}$ block of *replacing* choice probabilities $\Delta_{ij}(P)$ is given by:

$$\Delta_{ij}(P) = \begin{bmatrix} \Pi(j,1|i,1,P) & \dots & \Pi(j,\bar{a}-1|i,1,P) & \Pi(j,0|i,1,P) \\ \vdots & \ddots & \vdots & \vdots \\ \Pi(j,1|i,\bar{a}-1,P) & \dots & \Pi(j,\bar{a}-1|i,\bar{a}-1,P) & \Pi(j,0|i,\bar{a}-1,P) \\ \Pi(j,1|i,\bar{a},P) & \dots & \Pi(j,\bar{a}-1|i,\bar{a},P) & \Pi(j,0|i,\bar{a},P) \end{bmatrix}, \tag{25}$$

and the typical $\bar{a} \times \bar{a}$ block of *keeping* choice probabilities $\Lambda_i(P)$ is given by:

$$\Lambda_i(P) = \begin{bmatrix} \Pi(\kappa|i,1,P) & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \Pi(\kappa|i,\bar{a}-1,P) & 0 \\ 0 & \dots & 0 & 0 \end{bmatrix}. \tag{26}$$

In addition, the bottom row and the rightmost column blocks in (24) are given by:

$$\Delta_{\o j}(P) = \Big[\Pi(j,1|\o,P),\dots,\Pi(j,\bar{a}-1|\o,P),\Pi(j,0|\o,P)\Big], \quad \Delta_{i\o}(P) = \begin{bmatrix} \Pi(\o|i,1,P) \\ \vdots \\ \Pi(\o|i,\bar{a},P) \end{bmatrix}.$$

Each $\bar{a} \times \bar{a}$ block in the trade transition probability matrix $\Omega(P)$ refers to the cars of each make/model. The trade probabilities $\Pi(j,d|i,a,P)$ are strictly positive for non degenerate GEV distributions for ϵ, form the bulk of the interior of $\Omega(P)$, and the probabilities of keeping $\Pi(\kappa|i,a,P)$ appear on the diagonal. The bottom row contains the probability of buying a car $\Pi(j,d|\o,P)$ by households who don't have one. The last column contains the probability $\Pi(\o|i,a,P)$ of choosing the no car state, and the bottom left corner element is the probability of remaining in the no car state $\Pi(\o|\o,P)$.

Note that because the cars of the terminal age $\bar{a}$ cannot be traded, we use the last column of each block $\Delta_{ij}(P)$, the sub-transition probabilities for trading car i for car j, to hold the choice probabilities $\Pi(j,0|i,a,P)$ corresponding to buying a new car of type j. Similarly, the last diagonal element in the each block $\Lambda_i(P)$ is zero because the cars of age $\bar{a}$ can not be kept, and instead have to be scrapped during the trading phase.

The structure of the trade transition probability matrix $\Omega(P)$ corresponds to the block structure of the ownership distribution q in (20). The matrix product $q\Omega(P)$

represents the distribution of car ownership in the economy after the trading phase, assuming that all demand is satisfied (which is true in equilibrium). The result is the *post-trade* holdings distribution $q\Omega(P)$, which reflects the distribution of car holdings after the instantaneous trading phase has occurred where new cars are delivered to households who demand them, and used cars which households choose to scrap (both endogenous and exogenous scrappage) are removed from the economy. Due to the special arrangement of the columns in the trade transition probability matrix, the elements in the post-trade holdings distribution are reordered such that the fraction of owners of new cars in $q\Omega(P)$ is the last element in each subvector of length $\bar{a}$.

After the instantaneous trading phase, households own and drive their cars and the aging and accidents in these cars is governed by the $(J\bar{a}+1) \times (J\bar{a}+1)$ block-diagonal stochastic matrix that we refer to as *physical transition probability matrix* Q

$$Q = \begin{bmatrix} Q_1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \\ 0 & \dots & Q_J & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}, \text{ where} \tag{27}$$

$$Q_j = \begin{bmatrix} 0 & 1-\alpha(j,1) & \dots & 0 & \alpha(j,1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1-\alpha(j,\bar{a}-2) & \alpha(j,\bar{a}-2) \\ 0 & 0 & \dots & 0 & 1 \\ 1-\alpha(j,0) & 0 & \dots & 0 & \alpha(j,0) \end{bmatrix}. \tag{28}$$

Each $\bar{a} \times \bar{a}$ block Q_j forms a transition probability matrix which governs the evolution of cars of make/model j. The first $\bar{a}-1$ rows of the matrix describe the joint effect of deterministic aging and stochastic exogenous scrappage. As described in the previous section, the latter is modeled as a direct transition to the terminal age $\bar{a}$ with probability $\alpha(j,d) \in [0,1)$, resulting in a compulsory scrappage next period. Cars of age $\bar{a}-1$ reach the terminal age $\bar{a}$ with certainty as both aging and exogenous scrappage lead to the same outcome $\bar{a}$. The last governs aging and accidents of new cars. Finally, the bottom right corner element of Q denotes the transition by households who choose the no-car state.

It follows immediately that the product of the post trade ownership distribution and the physical transition matrix matrix, $q\Omega(P)Q$, gives the ownership distribution in the beginning of the next period. It is then clear that the stationary ownership distribution is simply given by an invariant distribution of the matrix $\Omega(P)Q$, which the following theorem shows is unique for any P.

Theorem 1. *Let $\sigma \geq \sigma_r \geq \sigma_j \geq \sigma_s > 0$. Then for any vector of prices P there is a unique invariant distribution q that satisfies the stationarity condition $q = q\Omega(P)Q$, and is a continuously differentiable function of P.*

Proof. See Appendix A.1 (page 64). □

Uniqueness of the stationary ownership distribution in Theorem 1 simply follows from the fundamental theorem of Markov chains, once we realize that with positive GEV scale parameters the choice probabilities have full support, and therefore transition matrix $\Omega(P)Q$ is irreducible and aperiodic. However, to show differentiability of $q(P)$ with respect to P we need a more involved argument given in the Appendix.

We are now in position to formally define and prove existence of equilibrium in the automobile market that extends the stationary flow equilibrium concept of Rust (1985a) to economies with positive transactions costs and discrete goods.

Definition D1 (Stationary equilibrium in the automobile market)**.** *A stationary equilibrium in the economy with a unit mass of consumers and cars of J makes/models and ages bounded above by $\bar{a}$, is given by the price vector and the ownership distribution probability vector $(P, q) \in \mathbb{R}^{J(\bar{a}-1)} \times \mathbb{R}^{J\bar{a}+1}$, such that the following conditions are satisfied:*

(a) Consumers follow their optimal trading strategies that arise from the solution of the dynamic problem (2)-(8);

(b) The market clearing conditions are satisfied: the excess demand is zero;

(c) The ownership distribution q is time invariant;

(d) New cars are supplied at fixed prices $\overline{P}_i$ and scrapped at prices $\underline{P}_i$, $i \in \{1, \ldots, J\}$, infinitely elastically.

Theorem 2. *The stationary equilibrium in the economy without persistent consumer heterogeneity defined in Definition D1 exists. In equilibrium the ownership distribution q satisfies the stationarity condition, and the equilibrium prices P satisfy the market* *clearing condition:*

$$q\Omega(P)Q = q, \tag{29}$$

$$ED(P, q) = 0. \tag{30}$$

Proof. See Appendix A.2 (page 65). □

Briefly, the proof uses Theorem 1 to guarantee that the equilibrium distribution q given by (29) is a smooth function of market prices P. The same is true for all major components of the model, namely the expected values EV which constitute the fixed point of the Bellman operator Γ in (9), all choice probabilities, and the excess demand $ED(P, q)$ given in (23). Then, given that the excess demand $ED(P, q)$ in our framework is bounded to the $(-1, 1)$ hypercube, we construct a continuous map satisfying the conditions of the Brouwer fixed point theorem, which establishes the existence of equilibrium. A number of intermediate results that the proof of Theorem 2 relies on, and turn out to be very useful for our computational framework, are formulated as separate lemmas in Appendix A.

It follows directly from the stationarity of the ownership distribution q that the fraction of population without cars is also time-invariant. Algebraically it can be seen from comparing the last elements in the left and right hand sides of the stationarity condition (29). Because the last column in Q has only one non-zero element, it follows that

$$\sum_{i=1}^{J} \sum_{a=1}^{\bar{a}} \Pi(\o|i, a, P) q_{ia} + \Pi(\o|\o, P) q_{\o} = q_{\o}. \tag{31}$$

Thus, a consequence of the stationarity of the equilibrium is that the fraction of consumers who demand the outside good, i.e. choose not to have a car, in the left hand side of (31) equals the “supply the outside good”.

Another consequence of Definition D1 and the two conditions in Theorem 2 is that the economy is in *stationary flow equilibrium*, i.e. it exhibits the *steady flow* property that the outflow of cars due to endogenous and exogenous scrappage equals the inflow of new cars of each make/model $j \in \{1, \dots, J\}$. If the economy were not in a stationary flow equilibrium there would either be a continual increase or decrease in the total stock of cars of each type j in the economy over time.

Theorem 3. *In a stationary equilibrium under the conditions of Theorem 2 the steady flow property is satisfied for each car make/model $j \in \{1, \dots, J\}$*

$$\underbrace{\sum_{a=1}^{\bar{a}-1} \Pi(1_s|j,a,P)\big(1-\Pi(\kappa|j,a,P)\big)q_{ja} + q_{j\bar{a}}}_{\text{outflow of scrapped cars of make/model } j} = \underbrace{\sum_{i=1}^{J}\sum_{a=1}^{\bar{a}} \Pi(j,0|i,a,P)q_{ia} + \Pi(j,0|\o,P)q_{\o}}_{\text{inflow of new cars of make/model } j}. \tag{32}$$

Proof. See Appendix A.3 (page 65). □

With the result of Theorem 3, it is clear that Definition D1 defines a *stationary flow equilibrium* in the market of new and used cars. Note that Theorem 2 only establishes the existence of the stationary flow equilibrium, but is silent about its uniqueness. Uniqueness of the equilibrium ownership distribution under any market prices P is established by Theorem 1, however, we have been unable to find high level conditions that guarantee uniqueness of the equilibrium price vector P. Despite this, we have computed many equilibrium solutions and have never encountered an issue of multiplicity of equilibria for a variety of utility function specifications and parameter values. Thus, we conjecture that there are conditions under which a stationary flow equilibrium not only exists, but is unique.

3.5 Numerical implementation

The key to success for our numerical implementation is the possibility to use the efficient Newton-based methods for finding the fixed point of the smooth Bellman operator Γ in the dynamic programming part of the model (9), and when solving the nonlinear system of equations (30) to find the equilibrium price vector P.

As is well known, Newton's method has a quadratic convergence rate when initiated from a sufficiently close starting point in a domain of attraction of the solution. In the dynamic programming part of the algorithm we rely on the globally convergent method of successive approximations before switching to Newton-Kantorovich iterations, in the same way it is done in the nested fixed point estimator (NFXP) in Rust (1994). In the equilibrium price search we initialize the Newton solver at the equilibrium prices of a similar model without consumer heterogeneity and transaction costs that can be computed as a solution to a system of linear equations as shown in Appendix B.

Appendix A contains several lemmas that establish the prerequisite smoothness properties, and specify the analytical formulas for the required gradients. We first show that EV is a smooth function of P implicitly defined by the fixed point condition, $EV = \Gamma(EV, P)$. It follows via chain rule that all value functions and all choice probabilities are also continuously differentiable in P. Differentiability of excess demand function $ED(P, q)$ in P for any q immediately follows.

Uniqueness and differentiability of $q(P)$ as an implicit function of P is established by Theorem 1. Using the chain rule, we obtain the Jacobian matrix for $ED\big(P, q(P)\big)$, and solve the market clearing conditions $ED\big(P, q(P)\big) = 0$ as a system of $J(\bar{a} - 1)$ non-linear equations in prices. We can use Newton's method for this, but also using the chain rule to compute the *total derivative* of $ED(P, q(P))$ with respect to P, which for lack of better notation we denote by $\nabla_P ED(P, q(P))$. The computational algorithm involves the following steps:

1. For a given vector of market prices P, solve the Bellman equation (9) for the fixed point $EV(P)$ using Newton-Kantorovich iterations;
2. Compute choice probabilities and form the trading and physical transition probability matrices $\Omega(P)$ and Q;
3. Compute the ownership distribution $q(P)$ as an invariant distribution of $\Omega(P)Q$;
4. Calculate excess demand and update prices via Newton's method
   $P' = P - \, [\nabla_P ED(P, q(P))]^{-1} \; ED(P, q(P))$;
5. Exit if convergence criterion for $ED\big(P', q(P')\big) = 0$ is satisfied, otherwise replace P by P' and return to step 1.

Thus, it is possible to use the gradient-based Newton's method in all steps of our numerical implementation, resulting in a fast algorithm for computing the stationary flow equilibrium in the automobile market.[22] Given how quickly the equilibria can be computed for various parameter values and specifications of the model, it can be nested within other algorithms such as a maximum likelihood estimator that we develop in Section 5.

[22]Implementation code will be available through a public repository.

4 Equilibrium with Persistent Consumer Heterogeneity

The economy with idiosyncratically heterogeneous households analyzed in the previous section is sufficient to support trade in presence of transactions costs. However, when we allow for more persistent consumer heterogeneity there are even larger gains from trade that result in equilibrium sorting of different ages of cars by different types of households.

4.1 Time invariant heterogeneity

We start with the superficially opposite case to the idiosyncratic heterogeneity considered in the previous section, namely the economy with permanent household types. However, the key to tractability of our framework is that we introduce persistent heterogeneity *in addition* to the idiosyncratic heterogeneity due to GEV random components. Later we add an intermediate form of time varying heterogeneity as well, which in the end gives us the best of both worlds: more realistic flexible forms of consumer heterogeneity, while retaining the elegance and computational tractability from the nested logit GEV specification introduced in the previous section. It is essentially a form of "mixed logit" that has been so useful in empirical work.

We introduce the symbol $\tau \in \{1, \dots, N_\tau\}$ to denote households of different permanent *types*, and denote f_τ the fraction of households of type τ. The structure of the household decision problem (2)-(5) is identical for all households, but due to arbitrary differences in preferences the solution becomes type-specific. Demand and supply from all types are aggregated in market equilibrium, and households endogenously specialize in their holdings of different makes/models and ages of automobiles in response to market prices and differences in their preferences. We allow for essentially unlimited flexibility in how the preferences of households of different types τ differ, as long as each type conforms to the general structure introduced in Section 3.1.

Let $u_\tau(i, a)$ be the utility for owning a car of make/model i and age a by households of type τ. Solving the Bellman equations (2)-(5) N_τ times we obtain the decision-specific value functions $v_\tau(i, a, j, d, s)$, expected value functions $EV_\tau(j, d)$, and choice probabilities $\Pi_\tau(j, d, s|i, a, P)$ for each household type τ, for all states $a \in \{1, \dots, \bar{a}, \o\}$, $i \in \{1, \dots, J\}$ and for all choices in $C(i, a)$ defined in (1).

Denote q_τ the ownership distribution of type τ households which we define similarly to (20) as a proper stochastic vector in $\mathbb{R}^{J\bar{a}+1}$. The full ownership distribution in the economy can be written as $q = (q_1 f_1, \ldots, q_{N_\tau} f_{N_\tau}) \in \mathbb{R}^{N_\tau(J\bar{a}+1)}$, but it is sufficient to work with its type-specific subvectors. Repeating the definitions of supply and demand from Section 3.4, we can derive τ type-specific excess demand functions $ED_\tau(P, q_\tau)$.

To extended Definition D1 for stationary equilibrium to the economy with permanent household types, note that conditions (a) and (d) remain unchanged, and we only have to modify the market clearing and stationarity conditions. Bearing in mind that trade is allowed between household types, $ED_\tau(P, q_\tau)$ does not need to be zero for each τ, instead the *integrated* demand has to clear, leading to the following condition

$$ED(P, q) = \sum_{\tau=1}^{N_\tau} ED_\tau(P, q_\tau) f_\tau = 0. \tag{33}$$

Further, with multiple types of households we require stationarity of the ownership distribution for each household type

$$q_\tau = q_\tau \Omega_\tau(P) Q_\tau, \ \forall \tau, \tag{34}$$

and thus stationarity of the ownership distribution in the whole economy. By making the aging transition probability matrix Q_τ household type specific in (34), we allow scrappage probabilities to vary by household type.

Theorem 4. *The stationary equilibrium in the economy with $\tau \in \{1, \ldots, N_\tau\}$ time invariant household types in addition to idiosyncratic heterogeneity, see Definition D1, exists. In equilibrium the ownership distribution $q \in \mathbb{R}^{N_\tau(J\bar{a}+1)}$ is composed of type shares weighted subvectors q_τ, each of which satisfies the stationarity condition* (34)*, and equilibrium prices P satisfy the market clearing condition* (33)*. Steady flow property of the equilibrium continues to hold.*

We omit the proof of Theorem 4 because it follows from straightforward modifications of the proofs of Theorems 1, 2 and 3 of Section 3. Fully detailed proofs are available on request. But to provide a rough idea of how the proof works, first note that we can use Theorem 1 to prove that for each consumer type τ there is a unique invariant distribution $q_\tau = q_\tau \Omega_\tau(P) Q_\tau$ and this q_τ is continuously differentiable function of P. Then it follows from Lemma L2 that $ED(P, q)$ given in equation (33) is a smooth function of P. Then following the proof of Theorem 2 we can appeal to the Brouwer Fixed Point Theorem to prove the existence of an equilibrium with persistent heterogeneity. Finally, the steady flow equilibrium condition must also hold (otherwise the stock of cars would be continually increasing or decreasing over time) and this result can be proven via a proof similar to that for Theorem 3.

Overall, the stationary equilibrium is exactly as described in Section 3: the only additional step is aggregation of τ-specific excess demands. Otherwise, a price vector P sets excess demand to zero per equation (33) subject to the constraint that the ownership distributions for all types τ are stationary per equation (34). Moreover, most of the theoretical results from Section 3 apply directly for each household type, one by one with the key exception that excess demand need not be zero type by type, i.e. $ED_\tau(P, q_\tau)$ may not necessarily equal zero for each type τ even though aggregate excess demand must be zero, $ED(P, q) = 0$.

The computational approach from Section 3.5 does not change much at all: we compute the equilibrium by first solving equation (34) for $q_\tau(P)$ which is a smooth implicit function of P, and repeat this calculation N_τ times for every τ. Then the functions $q_\tau(P)$ are jointly substituted into the excess demand, and the corresponding non-linear system of equations in prices is solved, again with Newton's method. Therefore, as one part of the solution algorithm is repeated for each household type, and the other does not depend on N_τ, we conclude that the solver is only linearly more computationally costly.

4.2 Time varying and hybrid heterogeneity

Now consider the case of time varying types. Consider an exogenous Markov process with state space $\mathcal{Y}$ and transition density $\rho(y_{t+1}|y_t)$ for some time varying variable y_t that is household-specific, e.g. income, and which evolves independently for each household. Assuming y enters the utility for cars, $u(j, a, y)$ or the marginal utility of money, $\mu(y)$, the dynamic problem becomes more complex since the household now has to account for stochastic variation in the y_t state variable when considering his/her optimal car trading strategy. An unexpected negative income shock may induce the household to keep their older car and delay replacement, or conversely a positive income shock may induce them to trade their existing car and buy a new one, or upgrade to a different car make/model.

The Bellman equations (2)-(7), which describe the optimal trading strategy, needs to be altered to account for the extra state variable y_t, and need to include an extra integration with respect to the transition density $\rho(y_{t+1}|y_t)$.[23] Assume that $\{y_t\}$ is ergodic and has an invariant distribution $\lambda(y)$ satisfying

$$\lambda(y') = \int_{\mathcal{Y}} \rho(y'|y)\lambda(y)dy. \tag{35}$$

In the case when $\{y_t\}$ is a finite state Markov chain instead of a continuous state process, ρ is simply a transition probability matrix, and (35) can be written as $\lambda = \lambda\rho$, where λ is a probability distribution vector completely analogous to the distribution of permanent types $(f_1, \ldots, f_{N_\tau})$ in the previous section.

With time-varying heterogeneity, the value functions, choice probabilities and ownership transition probability matrices are indexed by y similar to the way they were indexed by τ in the time invariant case. Let q_y denote the ownership distribution conditional on y, which we again define similarly to (20) as a proper stochastic vector in $\mathbb{R}^{J\bar{a}+1}$. Its typical element q_{jay} is the share of households who own car type j of age a while in income state y.

Continuing with the analogy, let $ED_y(P, q_y)$ denote the excess demand function for a household whose income state is y. Though y changes over time for different households, there is a stationary cross-sectional distribution of y given by the invariant density $\lambda(y)$ defined above, and there is a stationary joint density of car ownership states and y given by $q_y(P)$. So, to extend the Definition D1 for stationary equilibrium to the economy with time variant household types, we modify the market clearing condition to

$$\int_{\mathcal{Y}} ED_y(P, q_y)\lambda(y)dy = 0, \tag{36}$$

which is still the system of $J(\bar{a} - 1)$ non-linear equations in prices and ownership distribution. However the latter is pinpointed by the modified stationarity condition that takes into account the stochastic evolution of types according to the transition density $\rho(y'|y)$, namely

$$q_{y'} = \int_{\mathcal{Y}} q_y \Omega_y(P) Q_y \rho(y'|y)\lambda(y)dy. \tag{37}$$

Theorem 5. *The stationary equilibrium in the economy with time varying household*

[23] Since these extensions are straightforward we omit the Bellman equations to save space.

types given by an exogenous ergodic Markov process $\{y_t\} \in \mathcal{Y}$ with transition density $\rho(y'|y)$ and stationary distribution $\lambda(y)$, defined in Definition D1, exists. In equilibrium the joint ownership-type distribution is given by the $\lambda(y)$ and q_y that satisfy the the stationarity condition (37), and the equilibrium prices P satisfy the market clearing condition (36). Steady flow property of the equilibrium continues to hold.

This proof is also very similar to the proof of Theorem 4 and will be omitted for brevity, though a full detailed proof is available on request. It is also possible to layer combinations of time-invariant and time-varying heterogeneity and these cases can be handled by combining Theorems 4 and 5. For example we could have a finite number of types τ with different transition densities $\rho_\tau(y'|y)$. We can extend the equilibrium conditions by integrating excess demand $ED_{\tau y}(P, q_{\tau y})$ over all y for each type τ and then sum over types. This requires computing stationary ownership distributions $q_{\tau y}$ for each (τ, y) combination using a τ-specific analogue of (37), and stationary distributions λ_τ for each time invariant type τ. We can then substitute these invariant distributions (taken as smooth implicit functions of P) into the formula for excess demand $ED_{\tau y}(P, q_{\tau y})$, and compute the equilibrium prices by searching for a vector P that solves the system of nonlinear equations

$$ED(P, q) = \sum_{\tau=1}^{N_\tau} f_\tau \int_{\mathcal{Y}} ED_{\tau y}(P, q_{\tau y})\lambda_\tau(y)dy = 0 \tag{38}$$

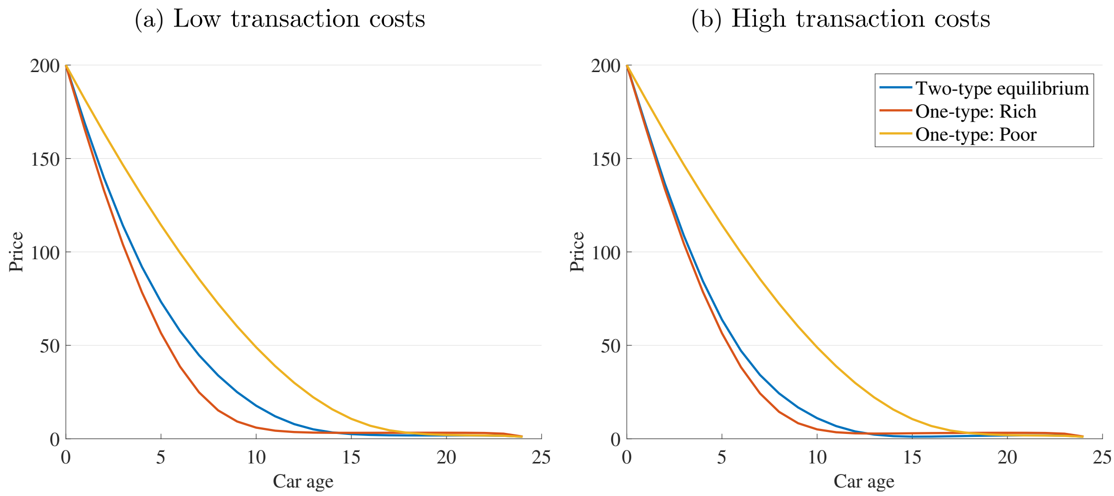
formed by integrating excess demand over all time-varying and invariant household types.

4.3 Illustrative example: sorting in stationary flow equilibrium

Figure 2 illustrates the stationary flow equilibrium in a heterogeneous agent economy with two permanent types of households who differ in their marginal utility of money, $\mu_2 = 0.3 > 0.1 = \mu_1$. The households who have a lower marginal utility of money are the rich households in this economy. The utility of the outside good is set to 0 for both households types. We assume this economy has 50% rich and 50% poor households.

In this example we collapse the GEV structure of random components to a simple extreme value EV1 distribution with common scale parameter $\sigma = 1$. Consumers also have the same discount factor $\beta = .95$ and the utility function is $u(a) = 10 - 0.5a$. There is a single car make/model $J = 1$ traded in this economy, with new car price $\overline{P} = 200$

Figure 2: Equilibrium price functions in a two household type economy



Notes: Both panels show the equilibrium price functions of the heterogeneous agent economy as well as homogeneous economies without transaction costs where all households are of either type.

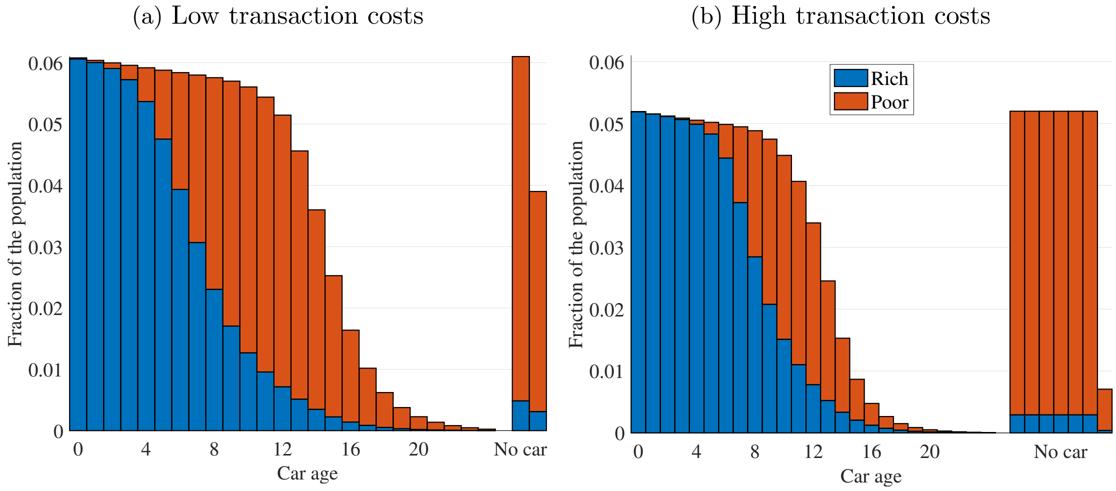
and scrap value $\underline{P} = 1$.

We also illustrate the effect of transactions costs by computing an equilibrium with buyer side transactions costs of $T_b(P, d) = 0$ and $T_s(P, a) = 0$ (low transaction cost), as well as an equilibrium with high transactions costs, $T_b(P, d) = 10$.

It seems reasonable to conjecture that in equilibrium a hand-me-down chain will emerge in which the rich are more likely to buy brand new cars whereas the poor households will buy the used cars previously owned by the rich. However it is not clear *a priori* what relative fractions of the two types of households will select into what fractions of the new and used cars, and which fraction choose to have no car at all. These questions and the effect of transaction costs on holdings can be answered by numerically computing the equilibrium prices and ownership distributions.

Figures 2 and 3, plots prices and holdings for two equilibria corresponding to the low and high transactions cost cases, respectively. For comparison, we also plot the prices that arise in the economies with single household types, one where all households have high marginal utility of money, and one where it is low. When transactions cost are high, equilibrium prices are closer to prices in the one-type economy where all consumers are rich because many poor consumers are now driven out of the market. Moreover, cars are scrapped earlier. Thus the higher transactions costs limits the gains from trade and partially "kills off" the market for used cars.

Figure 3: Equilibrium ownership in a two household type economy



Notes: The total area of the bars in each panel adds up to one, presenting both the distribution of households over car ages, and the share of outside good. There are equal numbers of households of each type. Both panels show the post-trade ownership distributions. Since the density in the no-car state is much higher than each car-by-age category, we have split the no-car bar into several bars placed horizontally next to each other.

There is clear evidence of sorting of households into the ages of the car in panel (a) of Figure 3, which shows the post-trade ownership distribution $q\Omega(P)$. The rich households hold the newer cars and in particular are much more likely to buy new cars than the poor households. In addition, the fraction of poor households who do not to own a car is much higher. Overall, we see that poor households are driven out of the market to a much larger extent than the rich households when transactions costs increase.

These findings confirm our conjecture that a "hand-me-down-chain" arises endogenously in equilibrium, created by type-specific *specialization in holdings* that facilitates gains from trade between the two types of households. The rich households buy brand new cars and hold them for several years and then sell them to poor households who also hold them for several years, trading the cars over a succession of poor owners until the car is scrapped. Thus, in this example the rich households are net suppliers of older cars, and the poor households are net demanders of older cars. Most of the trade between rich and poor households occurs for cars of roughly middle ages: the rich supply their middle aged cars to the poor households, and market clears in aggregate, but not for each type.

5 Identification and Structural Estimation

We have introduced a dynamic model of trade in automobiles that allows for rich specifications of observed and unobserved consumer heterogeneity whose equilibrium can be rapidly computed. By embedding the equilibrium solver into other algorithms we can therefore extend our framework in multiple directions including using the model in empirical applications. In this section we develop the *Doubly Nested Fixed Point* (DNFXP) algorithm to implement the maximum likelihood estimator for our modelling framework.[24] We also provide a discussion on model identification, including the case when the market prices are not observed.

5.1 Maximum Likelihood Estimation

Let θ be the vector of parameters characterizing consumer preferences, car transactions costs and car type specific accident rates that we wish to estimate. The solution of the model, including the equilibrium prices, quantities, and choice probabilities are then implicit functions of the parameter vector θ.

Suppose we observe types, states, choices and accidents for a random sample of consumers (households) indexed by $h \in \{1, \ldots, N_H\}$ where each consumer is observed over T_h separate time periods, which may or may not form a consecutive sequence. In other words we may have a balanced or unbalanced panel, or even a cross section if $T_h = 1$ for all h. Denote the total number of observations $N_{HT} = \sum_{h=1}^{H} T_h$. Let τ_h denote the observed type of consumer h.[25] Let x_{ht} denote the observed *pre-decision state* of consumer h in period t, so $x_{ht} = (i_{ht}, a_{ht})$ if the consumer owned a car of type i_{ht} and age a_{ht} at the start of period, *before making any decision about trading* and similarly $x_{ht} = \o$ if the consumer enters the period with the outside good. Let $(c_{ht}, s_{ht}) \in C(x_{ht})$ denote the consumer's decision from the choice set $C(\cdot)$ given in equation (1) where c_{ht} denotes the choice of whether to keep, trade or purge the current car and s_{ht} denotes the voluntary scrappage decision which is only relevant when $x_{nt} \neq \o$ and $c_{nt} \neq \kappa$. Finally, let $z_{ht} \in \{0, 1\}$ denote whether household h experienced a total loss accident leading to an exogenous scrappage

[24] Another natural extension of our framework is to include the equilibrium in the primary market of automobiles. However, given our empirical application in Section 6 to the Danish automobile market where all new cars are imported, we defer this extension to a future paper.

[25] We focus on the case of time invariant consumer heterogeneity. The cases of time varying and mixed consumer heterogeneity are straightforward generalizations.

of their car during period t. Note that given the structure of the state transitions, the next period state x_{ht+1} is fully determined by $(x_{ht}, c_{ht}, s_{ht}, z_{ht})$, and we can treat it as deterministic function of the current state and decision, $x_{ht+1} = f(x_{ht}, c_{ht}, z_{ht})$. This implies for any decision involving owning a car, $c_{ht} \neq \emptyset$, knowledge of (x_{ht}, c_{ht}) enables us to predict the probability of x_{ht+1} in terms of the accident probability $\alpha(c_{ht})$, so we can write a conditional density for (x_{ht+1}, s_{ht}) given x_{ht} as

$$\pi(x_{ht+1}|x_{ht}, \tau_h)\pi_s(s_{ht}|x_{ht}, \tau_h) \equiv \pi(f(x_{ht}, c_{ht}, z_{ht})|x_{ht}, \tau_h)\pi_s(s_{ht}|x_{ht}, \tau_h) = \\ \Big[I\{z_{ht} = 1\}\alpha(c_{ht}) + I\{z_{ht} = 0\}\big(1 - \alpha(c_{ht})\big)\Big]\Pi_{\tau_h}(c_{ht}|x_{ht})\Pi_{\tau_h}(s_{ht}|x_{ht}). \quad (39)$$

The probability $\pi(x_{t+1}|x_t, \tau)$ is simply an element of the matrix $\Omega_\tau(P)Q$ from the equilibrium condition (29) of Theorem 2 and the accident probabilities $\alpha(\cdot)$ are those used in the definition of the Q matrix in equation (27).[26] The conditional choice probability $\Pi(c_{ht}|x_{ht})\Pi_s(s_{ht}|x_{ht})$ on the right hand side of equation (39) are just the choice probability given in equations (13) and (18) of Section 3.3.

The (average) log-likelihood collects contributions from state transitions and the choice probabilities similar to Rust (1987):

$$L_H(\theta) = \frac{1}{N_{HT}} \sum_{h=1}^{H} \sum_{t=1}^{T_h} \left[\log\big(\pi(x_{ht}|x_{ht-1}, \tau_h, \theta)\big) + \log\big(\pi_s(s_{ht}|x_{ht-1}, \tau_h, \theta)\big)\right], \quad (40)$$

where the first term of $L_H(\theta)$ represents information from trading decisions, while the second term reflects information from scrappage decisions. We show below in Theorem 6 that the information on scrappage decisions is crucial for identification the key marginal utility of money parameters $\{\mu_\tau\}$ as well as transactions costs, $\{T_s(i,a), T_b(i,a)\}$ even when secondary prices P_{ia} are unobserved by econometrician. Thus, information on scrappage of cars is key to the identification of the model.

In applications with large sample size we can speed up the computation of the likelihood by avoiding the summation over h and t in (40), and instead use counts of similar observations. Let $N_{x'x\tau} = \sum_{h=1}^{H}\sum_{t=1}^{T_h} I\big\{x_{ht} = x', x_{ht} = x, \tau_h = \tau\big\}$ denote the total number of type τ consumers in state x who transit to state x'. We have $N_{HT} = \sum_\tau \sum_x \sum_{x'} N_{x'x\tau}$.

[26] Note that if $c_{ht} = \emptyset$, then the accident probability is not relevant and $\pi(x_{ht+1}|x_{ht})\pi_s(s_{ht}|x_{ht})$ reduces to $\Pi(c_{ht}|x_{ht})\Pi_s(s_{ht}|x_{ht})$, and if $x_{ht} = \emptyset$ or $c_{ht} = \kappa$, the scrappage indicator is not relevant, so we can define $\pi_s(s_{ht}|x_{ht}) = 1$ in such cases.

Similarly, let $N_{sx\tau} = \sum_{h=1}^{H} \sum_{t=1}^{T_h} I\{s_{ht} = s, x_{h\tau} = x, \tau_h = \tau\}$ be the total number of observations for scrappage choice $s \in \{0_s, 1_s\}$. In our empirical application to Denmark, H is the total number of households, and via using repeated cross sections of the entire population over $T = 12$ years, we end up with $N_{HT} = 39$ million observations. By the ergodic Law of Large Numbers we have with probability 1

$$\lim_{N_{HT} \to \infty} \frac{N_{x'x\tau}}{N_{HT}} = \pi(x'|x,\tau) q_\tau(x) f_\tau, \quad \lim_{N_{HT} \to \infty} \frac{N_{sx\tau}}{N_{HT}} = \pi_s(s|x,\tau) q_\tau(x) f_\tau. \tag{41}$$

Thus, the large sample limit of the cell-based likelihood function $L_H(\theta)$ is

$$\lim_{N_{HT} \to \infty} L_H(\theta) =$$
$$\sum_\tau \sum_x \left[\sum_{x'} \log\big(\pi(x'|x,\tau,\theta)\big) \pi(x'|x,\tau) + \sum_s \log\big(\pi_s(s|x,\tau,\theta)\big) \pi_s(s|x) \right] q_\tau(x) f_\tau, \tag{42}$$

so our data can be condensed into a much smaller number of empirical transition probabilities and market shares, $\{\pi(x'|x,\tau), \pi_s(s|x,\tau), q_\tau(x), f_\tau\}$, along with the consumer type distribution f_τ in the population. It is far faster to evaluate the likelihood in equation (42) than summing over 39 million individual observations as in equation (40).

In our empirical application in Section 6 we observe *scrappage outcomes* but not voluntary *scrappage decisions* because we do not observe accidents, including ones that lead to a forced scrappage of a vehicle. This implies that we cannot always observe the household's pre-decision state x_{ht} needed to evaluate the full information likelihood $L_N(\theta)$ given in equation (40) above. Nevertheless we show in Appendix C that we can still identify the accident probabilities $\alpha(i,a)$ for different car types and ages even when accidents are unobserved. We do this via a marginal likelihood function that integrates out with respect to accidents by calculating the probability of a scrappage outcome as the sum of two probabilities: 1) "exogenous scrappages" due to accidents that occur with probability $\alpha(i,a)$, and 2) "endogenous scrappages" due to a voluntary choice by the owner that occur with probability $(1-\alpha(i,a))\Pi_\tau(1_s|i,a)$. Technically the likelihood function can be expressed using a transition probability for the *post-decision state* augmented with the *scrappage outcome* which we denote by ζ_t, where $\zeta_t = 1$ if the car owned prior to trading is scrapped (either voluntarily or exogenously due to an accident), and $\zeta_t = 0$ if the car x_t was not scrapped. Define δ_t as the vector consisting of the post-decision state $c_t = (j_t, d_t)$

(i.e. the car the consumer has after the instantaneous trading decision) and the scrappage outcome ζ_t on the previous car $x_t = (i_t, a_t)$ (i.e. the pre-decision state). Thus we have $\delta_t = (c_t, \zeta_t)$ which equals $\delta_t = (j_t, d_t, \zeta_t)$ if the consumer traded car $x_t = (i_t, a_t)$ for car $c_t = (j_t, d_t)$ at time t and in our Danish data, δ_t is always observed.[27] In Appendix C we derive a transition probability $\pi(\delta'|\delta, \tau, \theta)$ and show that a likelihood function using these observed transition probabilities succeeds in identifying all parameters θ. This likelihood can also be written in both standard and cell forms as in equations (40) and (42).

Note that the formulation of the likelihood function above fully imposes the equilibrium constraints and rely on the equilibrium prices $P(\theta)$ implied by the model rather than observed prices at the secondary market. As show below, conditional on the model being well specified, these equilibrium constraints allow us to identify the structural parameters θ even when secondary prices P are unobserved. We can thus consistently estimate the true parameter $\theta^\star$ using the maximum likelihood estimator $\hat{\theta}$ defined by $\hat{\theta}_H = \arg\max_{\theta\in\Theta} L_H(\theta)$.[28]

5.2 Doubly nested fixed point (DNFXP) algorithm

To implement $\hat{\theta}$ we adopt a full solution approach similar to the Nested Fixed Point (NFXP) algorithm in Rust (1987), but where the additional equilibrium constraints require an additional nested loop to compute equilibrium prices. We refer to this as the *doubly nested fixed point* or DNFXP algorithm: while searching over the parameters space, DNFXP invokes the solution algorithm described in Sections 3 and 4 to compute the equilibrium prices, quantities, and choice probabilities necessary to compute the likelihood function. This requires an additional nested loop for calculation of the equilibrium prices P after the expected value functions $EV_\tau(P, \theta)$ and the corresponding choice probabilities are found as a result of solving consumers' dynamic programming problems.

While this seems like a daunting computational task, our extensive use of gradient based methods and principles of back propagation to compute them, makes it fast and

[27] If the pre-decision state is $x_t = \emptyset$ or the observed decision $c_t = \kappa$ then the scrappage outcome is not relevant, and δ_t simply equals x_t in such cases.

[28] When prices P are fully observed we do maximum likelihood subject to the constraint that $P(\theta) = P$ which results in a more efficient estimator due to the imposition of the constraint. However, even in situations where prices P are actually observed, researchers may not want to estimate subject to this constraint but rather rely on the model's ability to identify equilibrium prices using only micro data on car ownership transitions. The predicted prices $P(\hat{\theta})$ constitute some of the strong overidentifying restrictions of the model, so a Wald or Likelihood Ratio test of the hypothesis $H_0 : P(\theta^*) = P$ is likely to be a powerful test of the combined hypotheses of stationarity, individual optimality, and market equilibrium.

robust. For example, computing the gradient $\nabla_\theta L(\theta)$ of the likelihood function incorporates through the chain rules of calculus the computed gradient of the equilibrium prices $\nabla_\theta P(\theta)$, which in term incorporates $\nabla_\theta EV_\tau(P, \theta)$. The gradients needed at the outer loop (likelihood maximization) of the algorithms are produced as by-products of solving the model at the inner loops (equilibrium and Bellman optimality).[29] This enables us to use fast implementations of quasi-Newton algorithms such as the BHHH algorithm of Berndt, Hall, Hall and Hausman (1974) with accurate analytic gradients, without the need to compute the Hessian of $L(\theta)$ which is even more tedious. Together, this makes DNFXP feasible even on ordinary laptop computers.

5.3 Model Identification

In this section we establish the identification of the structural parameters θ of our equilibrium model under the *most unrestricted parametric* specification. Following the description in Sections 3 and 4, under the least restriction on the parameters of the model vector θ is composed of β and $\{u_\tau(i,a), \mu_\tau, T_b(i,a), T_s(i,a)\}$, for all $a \in \{0, \ldots, \bar{a}-1\}$, $i \in \{1, \ldots, J\}$, $\tau \in \{1, \ldots, N_\tau\}$, and thus has $1 + 2J(\bar{a}-1) + N_\tau(J\bar{a}+1)$ elements.[30]

In a theoretical analysis of identification the choice probabilities $\{\pi(x'|x,\tau), \pi_s(s|x,\tau)\}$ for each observed consumer type τ and the implied market shares $q_\tau(x)$ as well as type proportions f_τ as treated as known. The choice probabilities constitute the *reduced-form objects* of the model. Subject to an aribtrary location and scale normalization on utilities, the model is identified if only one vector of structural parameters $\theta^\star$ implies the reduced form probabilities.

A necessary condition for identification is that we have more observed "moments" (i.e. probabilities) than parameters being estimated. It is straightforward to show that we cannot identify the structural parameters using the aggregate market share data alone. This is simply due to the fact that such data provides only $J\bar{a}$ independent moments, but even after a location and scale normalization, the unrestricted specification for θ has far more parameters. Using data on consumer type specific market shares may suffice

[29]Because we extensively use the chain rule of calculus, our framework is compatible with many different specifications of preferences and can easily accommodate additional structural parameters. The computational cost of adding parameters is small — there is practically no additional time spent to compute additional derivatives, and so the run time for a single evaluation of likelihood function and its gradient hardly changes.

[30]The accident probabilities $\{\alpha(i,a)\}$ can also be treated as structural parameters as in our empirical application in Section 6.

depending on particular application. However, we have far more data than market shares in *ownership transitions* which constitute a total of $J\bar{a}(J\bar{a}+1)$ independent moments for each of the consumer types, and an additional $J(\bar{a}-1)$ moments for the age-specific scrappage probabilities for each car type. Thus with microdata or cell count data as described in Section 5.1 we typically have vastly many more moments than structural parameters in our model.[31]

There are two key reasons why we are able to identify μ_τ even when secondary prices P are unobserved. First, we do observe prices of new cars $\overline{P}_i$ as well as scrap values $\underline{P}_i$ for each make/model. Second, the quasi-linear structure of preferences together with the assumption that the market is in equilibrium imposes strong "cross equation restrictions" on car ownership transitions, holdings, and prices. In particular, $P_{ia}(\theta)$ is a nonlinear function of model parameters, so the price terms $\mu_\tau P_{ia}(\theta)$ and the car utilities $u_\tau(j,a)$ will not be collinear. Intuitively, we can identify μ_τ by observing where consumers endogenously "locate" in the "hand-me-down-chain" in the car market — richer consumers (those with lower μ_τ) are more likely to buy newer cars and more expensive brands, whereas poorer consumers are more likely to buy older cars and less expensive brands. Formally, identification of $\{\mu_\tau\}$ (up to a scale normalization) and transactions costs under additional identifying assumption is provided by the following theorem .

Theorem 6. *Provided that the market is in equilibrium and under the assumptions mostly laid out in Sections 3 and 4, namely*

(a) Random components ϵ of the utilities follow the GEV distribution,

(b) Prices of new cars $\overline{P}_i$ and scrap values $\underline{P}_i$ are observed for each car make/model i,

(c) Transaction cost of the new car buyer as well as transaction costs on the seller side for all $i \in \{1,\dots,J\}$, $a \in \{1,\dots,\bar{a}\}$ are included into the prices, $T_b(i,0) = T_s(i,a) = 0$,

the equilibrium prices P_{ia} and transactions costs $T_b(i,a)$ are point identified for all tradable cars $i \in \{1,\dots,J\}$, $a \in \{1\dots,\bar{a}-1\}$, and the marginal utilities of money μ_τ are point identified (up to a scale) for all consumer types $\tau \in \{1,\dots,N_\tau\}$.

Proof. See Appendix A.4 (page 66). □

[31] In the empirical analysis in Section 6 we allow for $N_\tau = 8$ (observed) household types, $J = 4$ car types, and assume $\bar{a} = 25$. This results in 1001 parameters in the unrestricted parameterization, which can not be estimated using only 96 independent market shares, or even 768 type-specific market shares. The total number of "moments" in the ownership state transitions is, however, 80800. Nevertheless, as described in Section 6 we actually use a parsimonious restricted specification with only 131 parameters.

The proof of Theorem 6 relies on the inversion theorem of Hotz and Miller (1993), namely the version of it for the GEV distributed shocks in Lemma 2 in Arcidiacono and Miller (2011). We can then treat the choice specific value function differences as data, and using the quasi-linear structure of the utility function are able to pick out its different parts by considering value contrasts for various pairs of choices. Under the assumption [c] that the seller side costs are "passed on" to the buyer all the monetary components in consumer preferences are identified.[32]

Once all the monetary components in consumer preferences are identified per Theorem 6, our model reverts to the standard dynamic discrete choice setting where the remaining structural parameters $\{\beta, u_\tau(i,a),\}$, $a \in \{0,\ldots,\bar{a}-1\}$, $i \in \{1,\ldots,J\}$ can be non-parametrically identified under the standard normalization and exclusion restrictions. In particular, proposition 2 in (Magnac and Thesmar, 2002) establishes identification of the utilities $u_\tau(i,a)$ assuming the discount factor β is known, and after location normalization by adding constants to all utilities so that $u_\tau(\emptyset) + \beta v_\tau(\emptyset,\emptyset) = 0$.

To identify the discount factor β we may resort to the exclusion restrictions shown in (Abbring and Daljord, 2020) to be sufficient for identification of the discount factor in the dynamic discrete choice models. An example of the appropriate exclusion restriction is the flatness of the utility function $u(i,a)$ for older cars such as $u_\tau(i,\bar{a}-2) = u_\tau(i,\bar{a}-1)$ for some i. Results of Abbring and Daljord (2020) then establish identification of the discount factor.[33]

Even though preferences are only identified up an arbitrary location and scale normalization, this can be sufficient to use the model to make counterfactual predictions of a wide number of policy changes, including changes to car taxation policy as we show in our analysis in section 6.[34] However we acknowledge that there are some counterfac-

[32] There are other types of identifying restrictions. In our empirical application in Section 6 we are able to identify transaction costs of both buyers and sellers even when accidents and secondary prices are unobserved under including driving and additional functional form restrictions on utilities.

[33] Theorem 6 does not pretend to provide the weakest possible conditions for identification, and identification can be established using alternative types of restrictions and incorporating other types of data not contemplated in that result. For example, in the model we actually estimate in section 6 we also use data on driving show how to extend the model to allow for driving. Since the marginal utility parameters μ_τ also affect observed driving, this is an additional source of identification. Identification can be assisted by placing stronger restrictions on households' utilities, $\{u_\tau(i,a)\}$ and in section 6 we assume these functions are quadratic in car age. With these additional restrictions (which are probably much stronger than absolutely necessary), we can identify all of the buyer side transactions costs (again assuming $T_s(i,a) = 0$ as in Theorem 6) and also the accident probabilities $\{\alpha(i,a)\}$ even when both accidents and secondary prices P are unobserved.

[34] See Kalouptsidi, Kitamura, Lima and Souza-Rodrigues (2021) who establish conditions where counterfactual predictions from dynamic single-agent discrete choice models are identified even though agent preferences are only partially identified.

tual predictions that cannot be identified without other information. For example, an upgrade to public transportation infrastructure would change the utility of the outside good, $u_\tau(\emptyset)$, but we are only able to identify how consumers evaluate the utility of cars *relative* to the outside good. We would need independent information on the incremental willingness to pay for an upgrade to public transport infrastructure in order to make counterfactual predictions of these sorts of policy changes.

5.4 Alternative estimation approaches

One drawback of the MLE approach to estimation is the requirement of *car ownership transition* data rather than the typical market share data that is commonly used to estimate vehicle choice models. In principle, we could use McFadden's method of simulated moments (MSM) (McFadden, 1989) to match the market shares in the outer loop of the DNFXP algorithm. To this extent we could use the stationary ownership distribution $q(\theta)$ or $q_\tau(\theta)$ implied by the model equilibrium to match the observed aggregate or consumer type specific market shares. However, as discussed in the previous section market share data alone are generally insufficient to identify the structural parameters without stronger functional form assumptions or utilizing additional data.

It follows from the identification argument in previous section that our method can estimate models with agnostic and flexible specifications of consumer preferences. Moreover, DNFXP approach allows for *direct modelling of price endogeneity* by capturing the functional dependence of equilibrium prices on heterogeneous consumer preferences for the observed and unobserved car characteristics. This contrasts our method from the well-known BLP method (Berry, Levinsohn and Pakes, 1995) which numerically inverts the mapping from expected discounted utility of different trading decisions (i, a) to its aggregate market shares q_{ia} and then uses the method of instrumental variables to regress the inverted market shares to the prices P which are endogenous right hand side variables. In contrast, we offer an *instrument-free* estimation approach.[35]

As we shall see in the next section, our model delivers *predictions* of secondary market prices and accident rates that are quite reasonable. Thus, direct full information maximum likelihood estimation with nested numerical solution of the equilibrium for each trial

[35]We are not the first to use an instrument-free full solution likelihood-based approach to identify and estimate the structural parameters. Yang, Chen and Allenby (2003) used a Bayesian approach for inference of the structural parameters of a static equilibrium model of simultaneous supply and demand.

value of the parameter vector θ enables us to overcome serious econometric challenges that incomplete estimation approaches such as BLP are unable to deal with.[36]

6 Analysis of Danish Car Tax Policy

In this section we use the DNFXP maximum likelihood estimator to structurally estimate a version of our model with 8 types of households and 4 different car types of 25 ages using a data set that follows the car holdings of all Danish households from 1997 to 2008 provided by Statistics Denmark. This data set contains nearly 39 million observations, yet we show that the 131 structural parameters of the model can be estimated in a matter of minutes using an ordinary laptop computer. We then show how the estimated model can be used to make counterfactual predictions that may be crucial for analysis of car tax policy in Denmark. We also extend the model by incorporating driving. This also enables us to study the effects of additional taxes such as fuel taxes and account for environmental and congestion impacts of hypothetical policy changes. We show that it is possible to raise tax revenue and consumer surplus while reducing CO_2 emissions by lowering registration taxes and raising fuel taxes.

We simulate the effects of reforms similar to ones that have been under consideration in Denmark, which shift taxation from the purchase of new cars to the use of cars by increasing the fuel tax. We compare the predictions from our estimated equilibrium model to those obtained from a model that does not account for equilibrium in the used car market, and instead assumes a proportional change in prices of new and used cars. Predictions that fail to account for equilibrium responses are unable to accurately capture behavioral responses to this policy change. They overestimate the change in fleet composition compared to an equilibrium analysis where used car prices, scrappage rates, and holdings of cars respond endogenously. Paradoxically, predictions from non-equilibrium models are too extreme, overpredicting tax revenue gains, for example.

[36] We refer to BLP as "incomplete" in the sense that it focuses on estimation of demand-side parameters (e.g. preferences for different cars) while avoiding nested numerical solution for equilibrium prices in order to directly model this endogeneity.

6.1 Incorporating Driving

The primary reason to own a car is to drive it and thus far we have ignored this important aspect of car ownership. Let x denote the number of kilometers a consumer chooses to travel in a period, and let p_j denote the price per kilometer traveled for a car of type j. This equals the price of fuel (e.g. kroner per liter) divided by the car's fuel efficiency (kilometers per liter of fuel) for car type j.

Let $u_\tau(j, a, x)$ be the utility a consumer of type τ obtains from owning a car of type j and age a and driving it (on average) for x kilometers during the period. We make a simplifying assumption that the probability of an accident and other physical deterioration in an automobile is independent of driving, x, and is instead only a function of car type j and car age a. The benefit of this assumption is that driving becomes *a static sub-problem* of the consumer's overall dynamic trading problem. The optimal amount of driving $x_\tau(j, a, p_j)$ then simply maximizes the driving utility net of monetary cost, $u_\tau(j, a, x) - \mu_\tau x p_j$. Substituting $x_\tau(j, a, p_j)$ back into the utility function $u_\tau(j, a, x)$ we obtain the *indirect utility* $u_\tau(j, a, p_j)$ for owning a car of age a that incorporates the individual's optimal choice of driving. Assuming that p_j is time-invariant, the resulting model falls within the specification of Section 3.

To allow for discrepancies between the theoretical optimal amount of driving and actual data on kilometers traveled by different cars between (bi-annual) inspections in Denmark, we treat $x_\tau(j, a, p_j)$ as *planned* driving by the consumer at the start of each period. *Actual* driving is subject to *ex post* unexpected events during the period that cannot be predicted exactly. We represent by ζ the net *ex post* effect of these unexpected driving needs on the marginal utility of driving, resulting in an *ex post* utility specification of the following form

$$u_\tau(j, a, x, \zeta) = \psi_\tau(j, a) + \big(\gamma_\tau(j, a) + \zeta\big)x - \tfrac{\phi_{\tau,j}}{2}x^2 - \mu_\tau p_j x, \quad \gamma_\tau(j, a) = \gamma_{\tau,j,0} + \gamma_{\tau,j,1}a, \tag{43}$$

where the first component does not depend on x and can be considered as the utility of owning a car apart from driving it, and therefore only affects trading behavior. We assume that $\psi_\tau(j, a)$ is a quadratic in the age of the car to fit the overall market shares across car types and ages, so we have $\psi_\tau(j, a) = \psi_{\tau,j,0} + \psi_{\tau,j,1}a + \psi_{\tau,j,2}a^2$.

The optimal *ex post* level of driving $x_\tau(j,a,p_j,\zeta)$ implied by this structure is:

$$x_\tau(j,a,p_j,\zeta) = \tfrac{1}{\phi_{\tau,j}}\big[-\mu_\tau p_j + \gamma_{\tau,j,0} + \gamma_{\tau,j,1}a + \zeta\big]. \tag{44}$$

Note that conditional on type-specific coefficient $\phi_{\tau,j}$, the parameters in (44) can be estimated by regressing the observed kilometers traveled to the cost of driving, household and vehicle characteristics. However, the regression only partially identifies a subset of structural parameters as ratios involving the parameter $\phi_{\tau,j}$, the coefficient governing the level of diminishing marginal utility from driving.[37]

When we substitute the expression for optimal *ex post* driving (44) back into the utility function (43) and take expectations over the *ex post* shocks ζ we obtain a specification for the *ex ante* indirect utility of car ownership that is a quadratic in age. By a slight redefinition, the parameters $(\gamma_{\tau,j,0},\gamma_{\tau,j,1},\gamma_{\tau,j,2})$ also subsume the first and second moments of the unobserved *ex post* shock to utility ζ. Thus, besides the marginal utility of money parameter μ_τ, there are a total of 6 unknown parameters for each consumer type τ and car type j, which are $\theta_{\tau,j} = (\psi_{\tau,j,0},\psi_{\tau,j,1},\psi_{\tau,j,2},\gamma_{\tau,j,0},\gamma_{\tau,j,1},\phi_{\tau,j})$. In Appendix D we show that the 6 parameters in $\theta_{\tau,j}$ are just-identified in terms of the 6 corresponding "semi-reduced-form" parameters, 3 for the linear driving equation, and 3 for the *ex ante* expected indirect utility of owning car (j,a) which is a quadratric in a after taking expectations of the *ex post* preference shock ζ: $E\{u_\tau(j,a,x(j,a,p_j,\zeta))\} = u_{\tau,j,0}+u_{\tau,j,1}a+u_{\tau,j,2}a^2$. This implies that we can estimate the model in two steps: first we estimate separate linear driving regressions for each (τ,j) combination to identify the 3 ratios $(-\mu_\tau/\phi_{\tau,j},\gamma_{\tau,j,0}/\phi_{\tau,j},\gamma_{\tau,j,1}/\phi_{\tau,j})$. Then we use the DNFXP algorithm to estimate and identify the 4 parameters $(\mu_\tau,u_{\tau,j,0},u_{\tau,j,1},u_{\tau,j,2})$ of the implied quadratic expected indirect utility function. Using these 7 estimated parameters, Appendix D shows that we can solve for all 7 of the structural parameters $(\mu_\tau,\psi_{\tau,j,0},\psi_{\tau,j,1},\psi_{\tau,j,2},\gamma_{\tau,j,0},\gamma_{\tau,j,1},\phi_{\tau,j})$. It is critical to fully identify all the underlying structural parameters in order to make counterfactual predictions involving changes in the fuel price paid by consumers, p, which in turn changes the per kilometer cost of driving, p_j, of the different types of cars $j \in \{1,\dots,J\}$.

[37]We ignore the restriction $x(j,a,p_j,\zeta) \geq 0$ which implies that ζ must be a truncated normal distribution.

6.2 Estimation Results and Model Fit

Though there are hundreds of different makes and models of cars sold in Denmark, for this analysis we aggregated them into 4 car types differentiated by their fuel economy and pollution levels ("green" for more fuel efficient, environmentally friendly cars and "brown" for others), and based on car weight ("heavy" versus "light"). We divided Danish households into 8 groups τ depending on whether they were a) singles or couples, b) whether the distance to work was short or long, and c) whether the household was rich or poor. The precise criteria for defining these groups are detailed in Appendix E.

We estimated a linear specification for preferences including driving as discussed in the previous section. Household preferences for cars decrease with age but at a diminishing rate, and there is heterogeneity in preferences for the different types of cars. We also estimated household-specific quasi-linear price sensitivity parameters μ_τ for each of the 8 types τ. By dividing the estimated coefficient $u_{\tau,j,0}$ for household τ's utility of a new car of type j by μ_τ, we obtain a measure of willingness to pay for one period's use of a new car in Danish kroner. For example we estimate that a rich couple with low work distance is willing to pay (e.g. rent for one year) a new light brown car for 36,704 DKK (or about US$5,580) compared to 32,682 DKK for the corresponding poor household. In general, we find that based on the revealed choices: 1) rich households are willing to pay more for any type of car compared to poor households, 2) all households preferred the heavy cars to the light ones and brown cars to green ones resulting in the following preference ordering: heavy brown $\succ$ heavy green $\succ$ light brown $\succ$ light green, 3) willingness to pay for cars by high work distance households exceeds that of low work distance ones, and 4) couples generally have higher willingness to pay for cars than singles. Given that there are a total of 131 parameters in the model we refer the reader to Appendix E for details on the maximum likelihood parameter estimates and standard errors.

The estimated model also has reasonable implications for driving (Appendix Table 4): households with high work distances drive much more than those with low, and more so for the rich. The estimated model implies fuel price elasticities between -0.10 and -0.60 across households. This is relatively close to Gillingham and Munk-Nielsen (2019), who find an average elasticity of -0.30 using a wide array of regression specifications.

The estimated equilibrium model provides a good fit to the observed distribution of car holdings and successfully captures key features of Danish households: 1) poor households are significantly more likely not to own a car than rich ones, 2) couples are more likely to own cars than singles, and 3) high work distance households are relatively more likely to own cars than those with low work distance, 4) a large fraction of households who are not car owners, 40%. The latter is in part due to the excellent public transport infrastructure and the widespread use of bicycles in Denmark, but also due to the high taxation of cars that we will analyze in more detail in the next section. Figure 4 shows how our model captures the post-trade age distribution of holdings of different cars by different households, including the "hand-me-down-chain" from rich to poor consumers. For example notice that for low work distance singles (the dark blue and red regions at the bottom of the bar graphs) the rich households (colored red) are relatively more concentrated in holding newer cars of each type whereas the poor households are more concentrated in holding older cars.

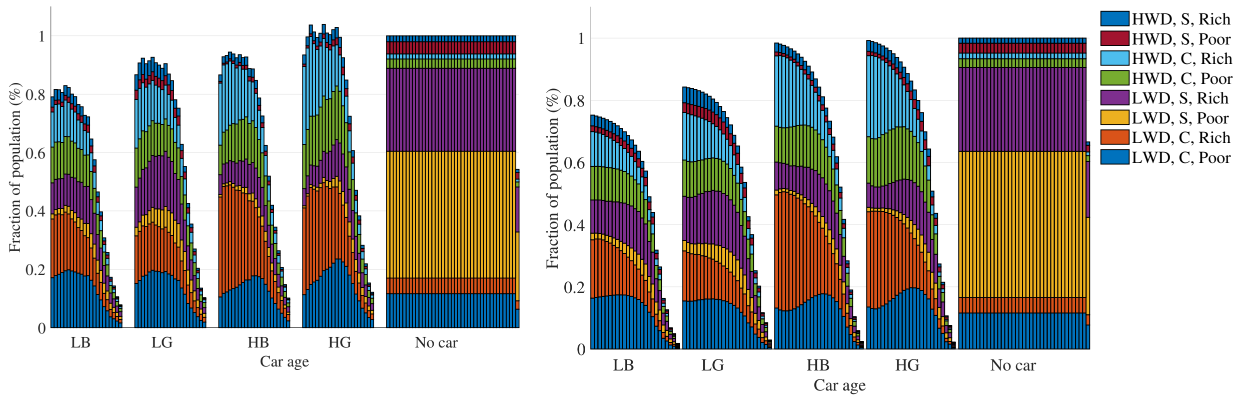
As we noted in section 5.1, our maximum likelihood estimation does not attempt to directly fit the holdings distributions, which we previously denoted by q_τ for each household type τ. Since Figure 4 plots the actual and predicted post-trade ownership distribution, $q\Omega(P)$, it also involves a comparison of the implied stationary distribution from our model, $q_\tau(\hat{\theta})$, to the non-parametric estimates q_τ from the data. Though our model slightly underpredicts holdings of new light cars and overpredicts holdings of new heavy cars, overall we think the model provides a remarkably good overall fit to over 800 non-targeted probabilities shown in figure 4 using a fairly parsimonious model with 131 parameters.[38]

As we noted above, the Danish register data contain information on nearly 80,000 car ownership transition probabilities, so the next figures provide some information on the model's ability to fit these transitions. Figure 5 illustrates the model's ability to capture the probability of cars purchases as well as the probability of keeping existing car. The left panel of Figure 5 plots the conditional probability that households purchase cars of a given age. The model closely tracks the observed purchase patterns at the aggregate level: households are more likely to buy a new car rather than any of the used ones and purchase probabilities declines as cars approaches the scrap age. When comparing these purchase probabilities for each of the 8 household types (results not shown), the model

[38]There are relatively few cars that are more than 20 years old in our data set and due to measurement issues relating to the oldest cars discussed in Appendix E we eliminated cars over 22 years old from our estimation sample. Thus the histograms for the data in figure 4 are truncated at age 22.

Figure 4: Actual and predicted holdings, by household type, car type and car age

(a) Actual holdings (b) Predicted holdings



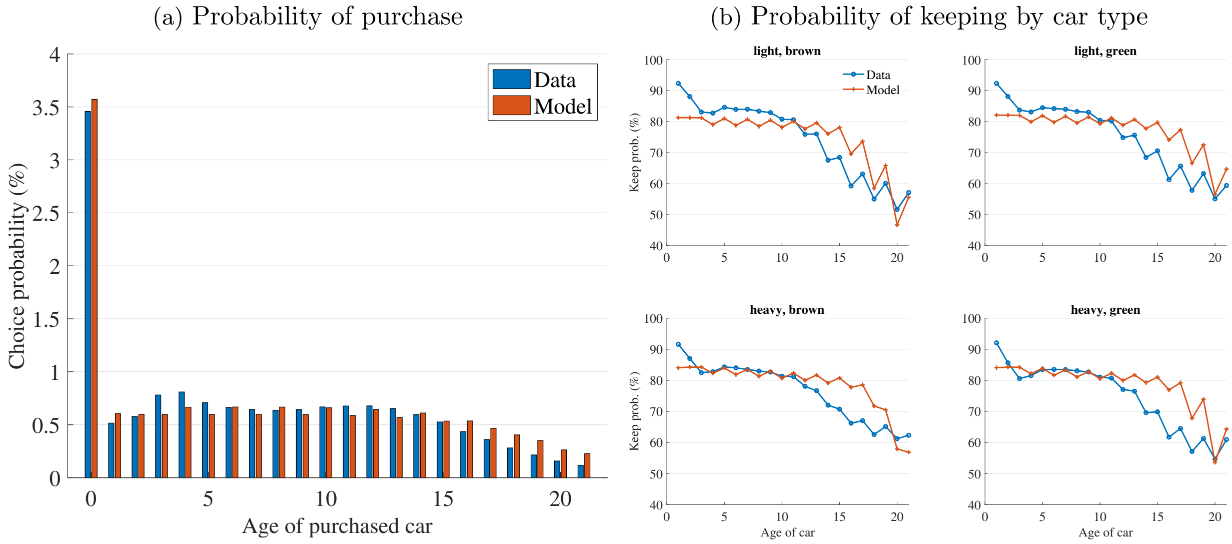
Note: The graphs show the fraction of the population holding each car-age combination as well as the outside good. For the outside good, we have opted to split the bars and put them next to each other since otherwise, that option would dominate the scale of the y axis. The abbreviations are: LB = light brown, LG = light green, HB = heavy brown, HG = heavy green. Within each of the four car types, car ages go from 0 to 24.

also closely tracks observed purchase probabilities and mimics the overall pattern from Figure 4 that rich, couple, high work distance households are the types most likely to buy a newer and larger (more expensive) cars.

The right panel of Figure 5 focuses on the 60% of Danish households that do own cars, and plots the conditional probability of keeping their existing as function of its age. The model is generally able to match that the overall level of probability of keeping a car, but also reveals an aspect of the data that our model is unable to capture well: we see that in the data, the probability of keeping a car is very high in the first couple of years and the gradually falls with the age of the existing car whereas our model predicts only a more modest decrease until it drops around age 15.

We conclude our presentation of the estimation results with Figure 6, which illustrate the model's predictions of quantities that we do not directly observe in our data set from Statistics Denmark. As we noted, our data allows us to observe *scrappage* of cars but not directly *accidents leading to scrappage* since another source of scrappages are *voluntary scrappages* by households, such as cars that are still drivable but may require expensive repairs to enable them to pass safety checks that a required before they can be sold to another household. As we described in section 5.1 we are able to identify and estimate accident probabilities using our structural estimation approach even though we do not observe accidents. The estimated parameters shown in Table 5 implies that accidents of

Figure 5: Actual and predicted probability of keep and purchase



the light cars are generally higher than the heavy ones, and accident rates rise quickly with age after cars are 15 years old, but are negligible when cars are new.

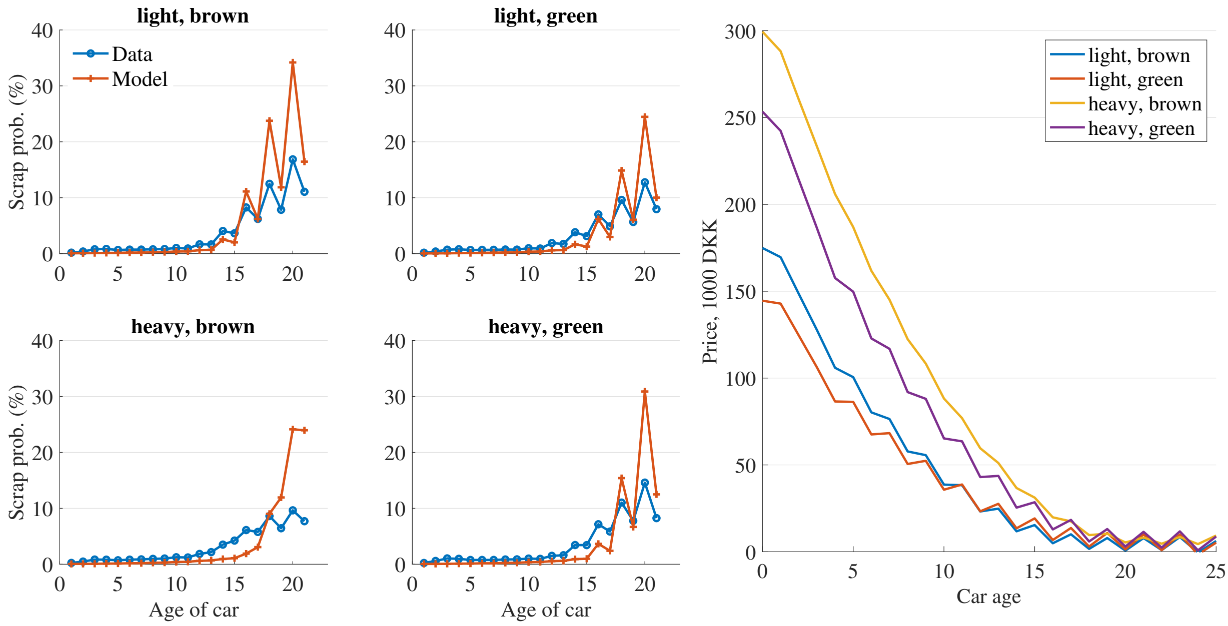
The left hand panel of Figure 6 displays the overall probability that a car is scrapped by the age and type of the car. We do directly observe when scrappages occur in the Danish Register Data so in this graph we can compare the model predictions (red lines) to the data (blue lines). Here we see the curious "zig-zag" pattern in scrappage rates that we noted in the introduction: Danish cars are much more likely to be scrapped at even ages than odd ages. We have verified that this effect is real and is not an artifact of how and when scrappages are recorded. It is due to the very strict biannual car inspections in Denmark that occur at even ages once cars are 4 years old or older. If the inspection reveals mechanical, safety or emissions problems, the owner is required to repair them in order to continue driving the car. We believe that Danes find these costs to be onerous and thus they are more likely to scrap rather then keep or sell their cars if they have problems when they are sufficiently old. We capture this effect in our model by including an even age dummy in the utilty of car ownership (skipping $a = 2$). The estimation results reveal big negative estimated coefficients for these dummy variables, with an estimated disutility that is typically 15% to 50% as large as the estimated single period utility the household obtains from owning a brand new car of the same type.

Finally, the right hand panel of Figure 6 shows the estimated secondary prices of

Figure 6: Zig-Zag Patterns in Scrappage and Predicted Equilibrium Prices

(a) Scrappage rates: data (blue) vs model (red)

(b) Predicted secondary prices



Note: Panel (a) shows the fit of the scrappage rates for each car and age (averaged over households weighted by the equilibrium ownership distribution), and panel (b) shows the equilibrium prices.

the 4 types of cars in our model. As we noted in section 5.1, even though we do not directly observe these prices, we can compute them for any trial value of the structural parameters, and use the substantial number of other moments in the data to identify both the structural parameters θ and the implied secondary prices, $P_{i,a}(\theta)$ for all four car types i and car ages a.[39] We firstly note that the rate of decline of our used car prices is broadly consistent with external evidence. We have limited data on suggested annual discount rates for used cars from the Danish Used Car Dealer Association, which suggest that prices should fall by 13% per year on average. Our model solution implies that prices fall on average 14% per year for three of the four cars, and 11% per year for the light green car. Thus, the overall magnitude of depreciation in our results is quite similar to the best data available.

Second, we note that the zig-zag pattern in scrappages is also present in the secondary prices predicted by our model. The effect on secondary market prices is a natural consequence of the estimated disutility our model predicts that Danish households experience

[39]We do observe prices of new cars, so we use the average new car prices $\overline{P}_i$ of the different makes and models in the 4 aggregate type groupings of cars as data rather than estimating these as additional parameters (see Appendix Table 3). The observed new car prices help to "tie down" secondary prices.

during the even aged years when their car is subject to inspection. As mentioned above, independent evidence from the limited data we have an actual used car transactions prices suggest that the zig-zag pattern in secondary prices that our model predicts is a real phenomenon in Denmark.

6.3 Counterfactual Policy Analysis

In this final section we carry out counterfactual predictions using our estimated equilibrium model of the Danish auto market. We focus on the effects of changes in the Danish new car (registration) tax and the fuel tax. As we noted in the introduction, Denmark has one of the highest new car taxes in the world. The tax is progressive with a rate of 105% of the retail price of the car in the first bracket (for the cars priced up to 81,000 DKK, approximately equal $16,000 USD at the time, excluding VAT), and a rate of 180% of the retail price in the second bracket. There is also a VAT of 25% applied to the wholesale price of a new car, prior to the calculation of the additional new car registration tax. Appendix E provides further details about car taxation policy in Denmark, which has been subject to vigorous political debate and a few reforms in recent years.[40]

In order to make the counterfactual predictions, which involve tax policy changes that affect new car prices $\overline{P}_i$ and thus used car prices $P_{i,a}$ as well fuel prices p that affect the price per kilometer driven for different car types, p_i, we need estimates of the "deep structural parameters" described in section 6.1 where we introduced driving into the model. These deep parameters are $\theta_\tau = (\mu_\tau, \{\psi_{\tau,j,0}, \psi_{\tau,j,1}, \psi_{\tau,j,2}, \gamma_{\tau,j,0}, \gamma_{\tau,j,1}, \phi_{\tau,j}\})$ that differ for each household type τ where the other parameters except the marginal utility of money μ_τ also differ by car type j. Recall the ψ parameters reflect the pure utility of ownership (independent of any driving) for different cars whereas the γ and ϕ parameters capture the utility from driving.

Once we have identified the deep structural parameters, we can systematically vary the registration tax rate (which affect the gross of tax new car prices $\overline{P}_i$ and thus also equilibrium secondary prices $P_{i,a}$) as well as the after-tax fuel price p which affects the cost per kilometer driven p_j of the different car types j. For each alternative policy we consider, we calculate the counterfactual equilibrium which also enables us to evaluate consumer

[40]After the end of our sample, the registration tax rate has been reduced across two separate reforms, in combination with changes to the treatment of electric vehicles (which were not present during our sample period).

welfare as well as the impact on overall tax revenues received by the Danish government. Specifically, we analyze a proposed policy involving cutting the new car registration tax in half, while increasing the fuel tax to offset the revenue loss from reduced new car taxes. Intuitively, this policy shifts taxation from the purchases of cars to their usage, with the intention to offset some of their harmful externalities. We assumed that the social cost of carbon is US $50/ton to estimate the marginal external social costs of driving, using results from Transport (2010) that also include negative externalities from congestion, accidents, noise and local air pollution.[41]

The overall purpose of this analysis is to illustrate the added value of using an equilibrium model to inform car tax policy. We want to illustrate how a policy maker would design a revenue-neutral reform that shifts taxation away from car registrations and towards fuel and usage. Specifically, we want to compare a sophisticated policy maker to a "naive" policy maker using a non-equilibrium model. As mentioned above, a naive way of handling the used car market in a non-equilibrium framework is to assume a proportional changes in used and new car prices. The baseline tax system for new cars is a two-part linear system with a kink, after which the marginal tax rate increases. Thus, the registration tax is progressive. We analyze a reform where both the low and high rates are cut in half.

We assume 100% passthrough of taxes to new car prices, which is consistent with our assumption in our estimation of perfectly elastic supply for new cars due to Denmark being a small open economy with no automobile manufacturing. We are not aware of any studies of passthrough in the Danish new car market, but full passthrough aligns with some studies in the U.S. new car market, such as Sallee (2011), but differs from others (Busse, Silva-Risso and Zettelmeyer, 2006). Assuming full passthrough to new car prices, cutting the low and high rates in half results in new car prices falling by between 25.6% and 27.6% for the four car types. In the non-equilibrium setting, which we refer to as *naive, expected,* we assume that the prices of the used cars of all vintages fall by the same percentages relative to the baseline equilibrium. However while it is easier to make predictions, the naive approach fails to account for the endogenous adjustment of

[41]See also Winston and Yan (2021) who analyze US data and find that traffic congestion can increase the demand for larger, less fuel efficient cars. Thus, congestion pricing, i.e. taxes or tolls that can mitigate congestion, "could reduce the vehicle fatality rate rate, generating $25 billion in annual benefits, and coule improve vehicle fleet fuel efficiency, generation roughly $10 billion in annual cost savings." (p. 196).

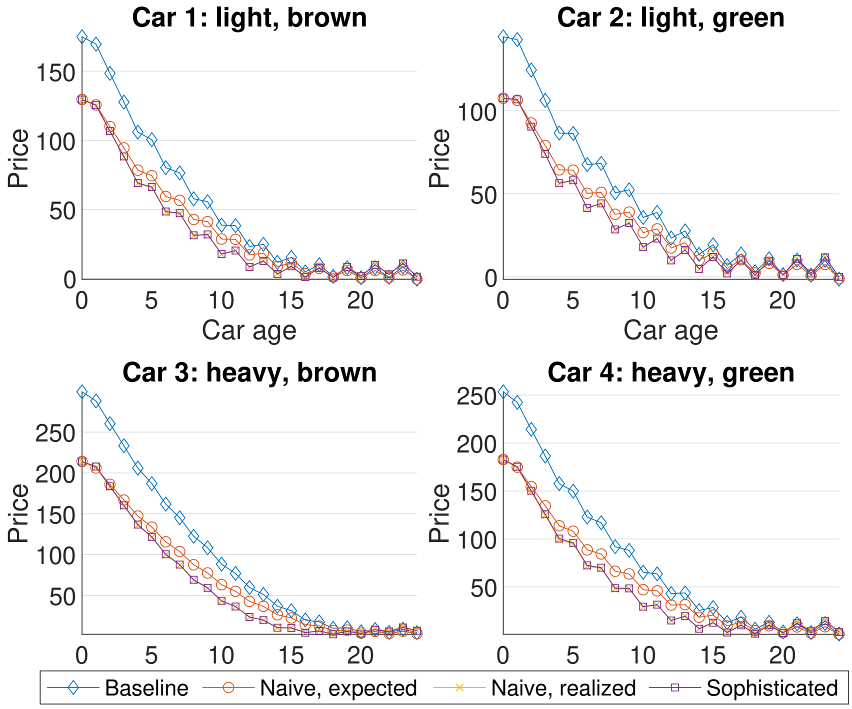
car trading and secondary prices to the change in fuel prices and new car prices. Thus, we will consider the following four scenarios:

1. **Baseline:** the model is solved for equilibrium prices and calibrated under *status quo* Danish tax policy as of 2008.
2. **Naive, Expected:** a naive policy maker, assuming that used-car prices will fall by the same proportion as the corresponding new car price for each car type. That is, the market is not in equilibrium. The policy maker raises fuel taxes until revenue is equivalent to the baseline. We calculate individual household welfare using these prices even though the used car market is not in equilibrium.
3. **Naive, Realized:** this is the equilibrium outcome that would result from the fuel tax policy enacted by the naive policy maker above. That is, the market is in equilibrium here and used car prices are set to equate supply and demand, but tax revenue is not equal to the baseline.
4. **Sophisticated:** these are the predictions of a sophisticated policy maker who correctly predicts the endogenous equilibrium responses to tax policy changes. That is, the used-car prices are such that the market is in equilibrium, and fuel taxes are set so that the total tax revenue is equal to the baseline tax policy scenario.

The outcomes under the four different policy scenarios are presented in Table 1, and the resulting car prices are in Figure 7. In the "Naive, expected" scenario, the policy maker is guided by a naive expectation of proportional passthrough. Lowering registration taxes results in an *increase* in tax revenue, so in order to achieve revenue equivalence, the policy maker increases fuel taxes from 57% of the price at the pump in the the baseline up to 76%.[42] According to this non-equilibrium model, that should achieve revenue equivalence at 9,391 DKK per household annually, but with a much younger car fleet where the average car age falls from 6.5 to 3.1 years. However, the scenario "Naive, realized" shows what will actually happen once used car prices adjust to equilibrate the market. Firstly, we note from Figure 7 that used car prices fall by *more* than proportionally, which correspondingly results in car ages falling by less than predicted, only to 4.3 years. In other words, the naive model predicts a much too strong movement towards newer cars, which results in excess demand for newer cars. As a result, registration tax

[42] Alternatively, the policy maker could also lower fuel taxes to achieve revenue equivalence. However, the required reduction implies a virtual abolishment of fuel taxes, which we judge to be less realistic in practice. Nevertheless, this illustrates the complexities in policy design with Laffer curve effects.

Figure 7: Equilibrium prices



Note: Each panel shows the prices for the corresponding car type where the four lines represent the four different scenarios (see Table 1 for descriptions).

revenue crumbles, resulting in total tax revenue falling short of the intended equivalence target from 9,391 to 7,452 DKK per household.

Instead, the column "Sophisticated" shows that the policy maker would only have to increase fuel taxes until they make up 73% of the price at the pump if she takes into account the endogenous responses in the used car market.

The general takeaway message is that a non-equilibrium model produces much greater movements in new car sales, and thus in registration taxes, than what an equilibrium model can sustain in flow equilibrium. This means that the policy maker will expect greater effects from changes in registration taxes than what will actually come to pass. The analysis so far has focused on comparing the decisions made under the guidance of a non-equilibrium as opposed to our equilibrium model. If we consider first the implications of the tax policy in the "Sophisticated" column, we note that the reform succeeds in raising total societal welfare. This is because although consumer surplus falls slightly from 7,364 to 5,969 DKK, driving-related externalities fall by more as total driving falls from 10,861 to 7,580 km annually in response to the much higher cost of driving. Thus,

Table 1: Policy Simulation Results

	Baseline	Naive, expected	Naive, realized	Sophisticated
Policy choice variables				
Registration tax (bottom rate)	1.050	0.525	0.525	0.525
Registration tax (top rate)	1.800	0.900	0.900	0.900
Fuel tax (share of pump price)	0.573	0.761	0.761	0.732
Exogeneous prices				
Price, light, brown (1000 DKK)	174.902	129.532	129.532	129.532
Price, light, green (1000 DKK)	144.551	107.532	107.532	107.532
Price, heavy, brown (1000 DKK)	299.452	214.048	214.048	214.048
Price, heavy, green (1000 DKK)	253.397	182.796	182.796	182.796
Fuel price (DKK/l)	8.322	14.885	14.885	13.243
Outcomes				
Social surplus (1000 DKK)	9.382	11.281	8.439	10.203
Total tax revenue (1000 DKK)	9.391	9.391	7.452	9.391
Fuel tax revenue (1000 DKK)	4.282	5.184	4.983	6.224
Car tax revenue (1000 DKK)	5.110	4.207	2.468	3.167
Non-CO2 externalities (1000 DKK)	6.751	3.385	3.281	4.711
Externalities (1000 DKK)	7.374	3.702	3.586	5.157
Consumer surplus (1000 DKK)	7.364	5.592	4.573	5.969
CO2 (ton)	2.148	1.094	1.052	1.537
VKT (1000 km)	10.861	5.446	5.279	7.580
E(car age)	6.507	3.080	4.336	5.417
Pr(no car)	0.367	0.535	0.534	0.418

Note: In the baseline scenario the institutional parameters conform to the data for 2008. In the column "Naive, expected", the two rates for the new car tax are both cut in half and then fuel taxes are increased until tax revenue is equal to the baseline. Used car prices are assumed to fall by the same percent as new car prices (i.e. 100% passthrough from the new to used car market). In "Naive, realized", the new car taxes and fuel taxes are as in "Naive, expected", but we solve for the equilibrium used car prices. In "Sophisticated", we change the fuel tax so that the total tax revenue is equal to the baseline, each time solving for the used car price equilibrium.

there are clear welfare differences between revenue equivalent combinations of the two tax rates on purchase and use.

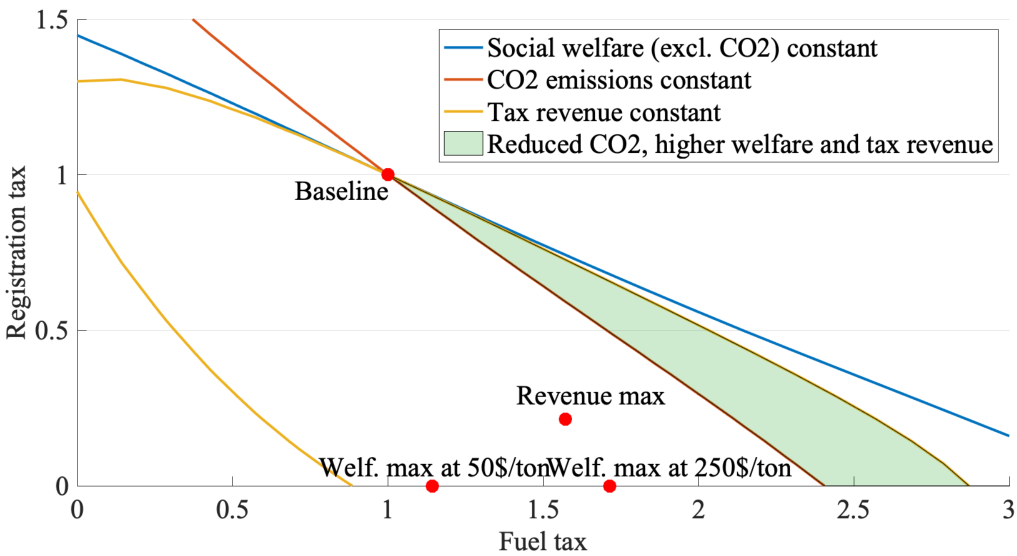
We next consider whether there are policy options that can raise welfare without harming tax revenue or increasing emissions. To do this, Figure 8 shows contour lines on a plot of the registration tax against the fuel tax.[43] The axes are scaled so that the point (1,1) denotes the baseline tax levels and the shaded green area to the southeast of the baseline levels, represent combinations of the two tax rates that result in lower emissions, higher welfare, and higher tax revenue–a "win-win-win" situation.[44]

[43]See Appendix E and specifically, Figure 9, for the 3D graphs these contours are taken from).

[44]The x and y axes show the tax rates for fuel and car registrations, respectively, normalized so that the baseline is 1. Each line represents the contour lines for one of three outcomes; that is, combinations of the two

Notably, the baseline (current situation) is quite far from the tax revenue-maximizing level, which has a much lower registration tax, but higher fuel tax. The welfare-maximizing tax combinations set the registration tax to zero, instead relying solely on the fuel tax to target externalities. The complex interaction between the two car taxes illustrates the importance of jointly modeling the purchase and driving decisions in an equilibrium framework, and the possibility of using such a model to optimize tax policy to meet policymaker objectives.

Figure 8: Welfare, Revenue and Environmental Effects of Different Tax Policies



7 Conclusion

We have introduced a computationally tractable model of equilibrium in the primary and secondary markets for automobiles that allows for flexible specifications of preferences and consumer heterogeneity and transactions costs. Our work was inspired by the early static discrete choice models of equilibrium in the automobile market pioneered by Manski,

tax rates where the outcome is kept constant and equal to the value in the baseline configuration, occurring at the point (1,1). The three outcomes are tax revenue, CO_2 emissions, and social welfare (excluding the external costs of CO_2 emissions). Moving from the baseline in the direction of the origin implies an *increase* in all three outcomes, although tax revenue will eventually start to decline again (because in the baseline, both tax rates are above the top point of the Laffer curve). Four points are depicted on the graph in red: first the baseline, (1,1). Second, the top point of the Laffer "curve", where overall tax revenue is maximized. And third, two points that show the overall social welfare maximizing policies: one under a CO_2 price of \$50/ton, and one at a higher price of \$250/ton (a price recently suggested by the Danish environmental council). Not surprisingly, a higher price results in a higher fuel tax, but still a zero tax on car purchases.

Sherman, and Berkovec and the subsequent efforts to extend their models to include dynamics and transactions costs and model equilibrium price setting in the primary market by Rust, Stolyarov, Gavazza, Lizzeri and Roketskiy, and Esteban and Shum. We believe that our framework is promising for empirical applications and policy analysis, and in future work we plan to further extend and apply it in a number of directions.

One of these directions is ongoing work (Gillingham, Iskhakov, Munk-Nielsen, Rust and Schjerning, 2019) to structurally estimate an "overlapping generations" version of our model using Danish register data to allow for a realistic counterfactual analysis of vehicle tax reform in Denmark. Another direction is to extend the model to include Bertrand-Nash equilibrium in the primary market for autos in addition to competitive equilibrium in the secondary markets. Estimation of the model with the primary market requires a triply nested version of NFXP, but the payoff is that we can use the model to relax our assumption of 100% passthrough of new car car taxes to retail prices, as well as use the model to predict merger counterfactuals.

Another direction would relax the assumption of stationarity and extend our definition of equilibrium to allow for macroeconomic shocks that can capture the pronounced "waves" often found in the age distribution of vehicles (Adda and Cooper, 2000). We are comparing different solution concepts in terms of computational tractability and empirical realism, including the "temporary equilibrium" concept of Grandmont (1977), the "sufficient statistic" approach of Krusell and Smith (1998), as well as a full blown rational expectations equilibrium that takes into account the entire holdings distribution of cars as a component of the "state variables" that consumers use to predict future prices as in Cao (2016).

A very challenging extension of our model would endogenize the characteristics of vehicles by allowing firms to invest in R&D to produce new vehicle designs. Longer-run competition on attributes will likely require a fundamentally non-stationary framework and raises questions of consumer expectations over future products. Very promising headway into this sort of analysis has been done in the pioneering work of Goettler and Gordon (2011), and it may be possible to adapt this approach into a more evolutionary model of the automobile market. A final challenging extension would be to incorporate asymmetric information in a more detailed treatment of the "microstructure" of trade in the automobile market, including endogenous intermediation of trade by car dealers as well as direct consumer transactions. Recent studies such as Biglaiser, Li, Murry and Zhou (2020) have provided new empirical insights into the microstructure of trade that are not modeled in our framework, but represent important directions to pursue in the development of more detailed and realistic models of the microstructure of trade in automobiles.

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Appendix

A Proofs

Lemma L1 (Jacobian Matrix of the Smoothed Bellman Operator)**.** *Let EV be the unique fixed point of the smoothed Bellman operator Γ in (9), and let $\nabla_{EV}\Gamma(EV,P)$ be the Jacobian matrix of Γ with respect to EV. The following holds:*

$$\nabla_{EV}\Gamma(EV,P) = \beta\Omega(P)Q, \tag{45}$$

where the matrices $\Omega(P)$ and Q are defined in equations (24) and (27), respectively. The norm of this matrix is $\|\nabla_{EV}\Gamma(EV,P)\| = \beta \in (0,1)$.

Proof. This Lemma can be proved via direct calculation, though the algebra involved is extremely tedious. We sketch a more conceptual proof of the result here. First, note that the operator Γ is a smooth nonlinear and recursively nested function of log-sum functions (11). The log-sum functions are in turn functions of the choice-specific value functions $v(i,a,j,d)$ given in equations (3), (6), and (8) of section 3. These value functions, call them v, are in turn functions of the expected values EV a result we emphasize by writing $\Gamma(EV,P) = \Gamma(v(EP),P)$. Using the chain rule to compute the Jacobian matrix $\nabla_{EV}\Gamma(EV,P)$ with respect to the $J\bar{a}+1\times 1$ vector EV, we have

$$\nabla_{EV}\Gamma(EV,P) = \nabla_v\Gamma(v(EV),P)\nabla_{EV}v(EV). \tag{46}$$

The Lemma follows by showing that $\nabla_v\Gamma(v(EV),P)$ equals $\Omega(P)$ and $\nabla_{EV}v(EV)$ equals βQ. The former result follows from the Williams-Daly-Zachary Theorem (see McFadden (1981)), and the fact that the Γ operator can be expressed as the expected maximum of the v functions with the additive GEV errors as shown via the representation of $\Gamma(EV,P)$ as the value functions V given in equations (2) and (5). The Williams-Daly-Zachary Theorem implies that the derivatives of the expected maximum of $v+\epsilon$ with respect to v equals the choice probabilities Π such as in equation (13). When the matrix of values v is arrayed in the same order that EV is arrayed (with first $J\bar{a}$ elements of EV equalling $EV(i,a)$ for $i=1,\ldots,J$ and $a=1,\ldots,\bar{a}$ and $J\bar{a}+1$-st element equal to $EV(\o)$ and we account for the fact that some elements of EV appear in multiple different elements in any given row of the matrix v), it is not difficult to see that $\nabla_v\Gamma(v(EV),P) = Q(P)$, where the latter is a $(J\bar{a}+1)\times(J\bar{a}+1)$ Markov transition probability matrix given in equation (24). Further using the formulas for $v(EV)$ in equations (3), (6), and (8) of section 3, it is not hard to see that $\nabla_{EV}v(EV) = \beta Q$, where Q is the accident/aging transition probability given in equation (27). Since Q and $\Omega(P)$ are both Markov transition probability matrices, so is their product, $M = \Omega(P)Q$. Recall the notion of a matrix norm, $\|M\| = \sup_{x\neq 0}\|Mx\|/\|x\|$ where $\|x\|$ is a norm of the vector x (e.g. Euclidean norm or sup-norm). When M is a transition probability matrix, it is easy to see that $\|Mx\| \leq \|x\|$, which implies that $\|M\| \leq 1$. Let e be a vector all of whose elements equal 1. Then $Me = e$ which implies $\|M\| \geq 1$, or $\|M\| = 1$. Also it is easy to see from the definition of a matrix-norm, $\|\beta M\| = \beta\|M\|$. It follows that $\|\beta\Omega(P)Q\| = \beta$. □

Lemma L2 (Differentiability)**.** *The unique fixed point EV of the smoothed Bellman operator in (9), the choice-specific value functions $v(\cdot)$ in (2), (5) and (7), the choice probabilities $\Pi(j,d,s|i,a)$ in (13), the trade transition probability matrix $\Omega(P)$ and excess* *demand function $ED(P)$ exist and are continuously differentiable functions of market prices P. The Jacobian matrix of the fixed point EV with respect to the market prices is given by*

$$\nabla_P EV(P) = -[I - \nabla_{EV}\Gamma(EV,P)]^{-1}\nabla_P\Gamma(EV,P), \tag{47}$$

where $\nabla_P\Gamma(EV,P)$ is the $J\bar{a}+1 \times J(\bar{a}-1)$ Jacobian matrix of Γ with respect to market prices P.

Proof. Existence of a unique fixed point $EV = \Gamma(EV,P)$ for any P follows because the operator Γ can be shown to be a *quasi-linear, monotone mapping* (see Rust, Traub and Wozniakowski (2002)) and thus is a contraction mapping with a unique fixed point EV. By Lemma L1, Γ is a continuously differentiable function of EV with gradient $\nabla_{EV}\Gamma(EV,P)$. By the Implicit Function Theorem we can express EV as a zero, $F(EV,P) = 0 = EV - \Gamma(EV,P)$ and the solution EV will be a continuously differentiable implicit function of P provided that $\nabla_{EV}F(EV,P)$ is invertible. However from Lemma L1 $\nabla_{EV}F(EV,P) = I - \beta\Omega(P)Q$, and this is invertible with the geometric series representation for its inverse

$$[I - \beta\Omega(P)Q]^{-1} = \sum_{t=0}^{\infty}\beta^t[\Omega(P)Q]^t. \tag{48}$$

Then since $EV(P)$ is a continuously differentiable function of P, it is easy to see from the formulas for v, the conditional choice probabilities Π and the transition probability matrix $\Omega(P)$ are continuously differentiable since they are explicit smooth functions of $EV(P)$. The formula for $\nabla_P EV(P)$ in equation (47) is a consequence of total differentiation of the identity $\Gamma(EV(P),P) = 0$ with respect to P and solving for $\nabla_P EV(P)$.

□

Lemma L3 (Gradient of invariant distribution)**.** *Consider a $n \times n$ Markov transition probability matrix $P(\theta)$ that depends on a parameter $\theta \in \mathbb{R}^k$ in a continuously differentiable fashion.*[45] *Let $h(\theta)$ be the unique invariant distribution of $P(\theta)$ satisfying the equation*

$$h(\theta) = h(\theta)P(\theta). \tag{49}$$

Then the $n \times 1$ transpose of $h(\theta)$, $h(\theta)'$, is the unique solution to the expanded $(n+1) \times (n+1)$ linear system given by

$$\begin{bmatrix} I - P(\theta)' & e \\ e' & 1 \end{bmatrix} \begin{bmatrix} h(\theta)' \\ 1 \end{bmatrix} = \begin{bmatrix} e \\ 2 \end{bmatrix} \tag{50}$$

where e is an $n \times 1$ vector all of whose elements equal 1, and I is a $n \times n$ identity matrix. Moreover, $h(\theta)$ is a continuously differentiable function of θ, and the simple expression for the Jacobian matrix $\nabla_\theta q(\theta)$ are readily available.

Proof. The stationarity condition (49) can be recast as $h(\theta)$ being a left zero of the matrix $I - P(\theta)$. The usual application of the Implicit Function Theorem to $F(h,\theta) = 0$ where $F(h,\theta) = h(\theta)[I - P(\theta)]$, would provide the result. Unfortunately the prerequisite

[45] Thus, we assume that the mapping $\nabla_\theta P(\theta)$ from R^k to R^{k*n*n} (where the latter can be interpreted as the space of k-tuples of $n \times n$ matrices) exists and is a continuous function of θ. To make things easier to understand, assume initially that $k = 1$ so we are considering $P(\theta)$ and $h(\theta)$ as functions of a single parameter θ. If θ has k components (i.e. $\theta \in R^k$) we simply "stack" the formulas we provide below in the univariate case into a k-tuple.

condition that $\nabla_h F(h,\theta)$ is non-singular at a zero of F fails as $\nabla_h F(h,\theta) = I - P(\theta)$, and this matrix is singular.

Let $A(\theta)$ be the $(n+1)\times(n+1)$ matrix on the left hand side of equation (50). It is not hard to show that $A(\theta)$ is invertible and we can write

$$\begin{bmatrix} h(\theta)' \\ 1 \end{bmatrix} = \begin{bmatrix} I - P(\theta)' & e \\ e' & 1 \end{bmatrix}^{-1} \begin{bmatrix} e \\ 2 \end{bmatrix}. \tag{51}$$

Then we have that $\nabla_\theta h(\theta)'$ is the upper left $n\times n$ submatrix of the product of $\nabla_\theta A^{-1}(\theta)$ times the vector $\begin{bmatrix} e \\ 2 \end{bmatrix}$. Further, the standard formula holds for the gradient of $A^{-1}(\theta)$ with respect to θ

$$\nabla_\theta A^{-1}(\theta) = -A^{-1}(\theta)\left[\nabla_\theta A(\theta)\right] A^{-1}(\theta), \tag{52}$$

and we have

$$\begin{bmatrix} \nabla_\theta h(\theta)' \\ 0 \end{bmatrix} = \begin{bmatrix} I - P(\theta)' & e \\ e' & 1 \end{bmatrix}^{-1} \begin{bmatrix} -\nabla_\theta P(\theta)' & 0 \\ 0' & 0 \end{bmatrix} \begin{bmatrix} I - P(\theta)' & e \\ e' & 1 \end{bmatrix}^{-1} \begin{bmatrix} e \\ 2 \end{bmatrix}. \tag{53}$$

□

Results of Lemma L3 are useful in both the differentiating the invariant ownership distribution with respect to the market prices for the equilibrium computations, and the structural parameters for the estimation of the model. In both cases, $P(\theta)$ is given by $\Omega_\tau(P)Q_\tau$ as per equation (34). Computing the required $\nabla_{P_{ia}}\Omega_\tau(P)Q_\tau$ for each price P_{ia} on the secondary market is straightforward. The same applies to all parameters of the utility function. Yet, if θ includes accident probabilities $\alpha(i,a)$ that enter in Q_τ, product rule has to be invoked leading to a slightly more involved expression $\nabla_{\alpha(i,a)}\Omega(\alpha)Q(\alpha) + \Omega(\alpha)\nabla_{\alpha(i,a)}Q(\alpha)$.

A.1 Proof of Theorem 1 (page 23)

Proof. When scale parameters of GEV distribution of random components ϵ are positive, $\sigma \geq \sigma_r \geq \sigma_j \geq \sigma_s > 0$, the choice probabilities are bounded away from zero for all choices and for any price vector P. Thus, the transition probability matrix $\Omega(P)Q$ is irreducible and aperiodic. Uniqueness of the stationary distribution q that satisfies $q = q\Omega(P)Q$ follows from the fundamental theorem of Markov chains.

To show continuous differentiability of the stationary distribution q given as an implicit function of P by $q = q\Omega(P)Q$, it would be enough to apply the Implicit Function Theorem, but unfortunately the prerequisite invertibility condition fails in our case. Indeed, q can be treated as a left zero of the matrix $I - \Omega(P)Q$, where I is the identity matrix of the appropriate size. In other words, $q(P)$ can be written as a zero the non-linear mapping $F(q,P) = q\big(I - \Omega(P)Q\big) = 0$. However when q is ergodic, Lemma L3 provides an explicit solution for $q(P)$ as the inverse of a bordered matrix for which $I - \Omega(P)Q$ is an upper left $(J\bar{a}+1)\times(J\bar{a}+1)$ submatrix. It also provides a formula for $\nabla_P q(P)$ in terms of gradients of the matrix $\Omega(P)$ with respect to P. This proves that $q(P)$ is uniquely defined and continuously differentiable function of P.

□

A.2 Proof of Theorem 2 (page 24)

Proof. The proof of existence of an equilibrium follows from Brouwer's Fixed Point Theorem by defining a mapping $\Psi(P) : R^{J(\bar{a}-1)} \to R^{J(\bar{a}-1)}$ where $\Psi(P) = P + ED(P)$, $J(\bar{a}-1)$ is the dimension of the price vector P and number of used cars traded in secondary markets with J types of cars sold in the primary market and $\bar{a}$ the oldest tradeable car age in each market, and $ED(P)$ is defined in (23). From Lemma L2 it follows that ED and thus Ψ is a continuous mapping from $R^{J(\bar{a}-1)} \to R^{J(\bar{a}-1)}$. Note also that for any P the components of $ED(P)$ lie in the interval $[-1, 1]$. Thus, when prices are sufficiently high, a vanishing number of consumers will wish to buy any new car but nearly all consumers will want to sell their cars, so $ED(P)$ will be close to a vector with all its components equal to -1. Similarly, for a sufficiently low set of prices (possibly negative), nearly all consumers will wish to buy used cars and very few will want to sell their vehicles at such low prices. So for such prices $ED(P)$ will be close to a vector with all of its components equal to $+1$. It follows that we can define a compact box B in $R^{J(\bar{a}-1)}$ where Ψ satisfies an "inward pointing" property on the boundaries of this box, so it follows that $\Psi : B \to B$. Since Ψ is a continuous mapping and B is a compact, convex set, the Brouwer fixed point theorem implies that a fixed point of Ψ exists, and it is clear that any such fixed point satisfies $ED(P) = 0$.

□

A.3 Proof of Theorem 3 (page 25)

Proof. The proof is based on the observation that the physical transition probability matrix Q in equation (27) is block diagonal, implying that the stationarity condition (29) in Theorem 2 can be written separately for each car make/model j as

$$\left(q \begin{bmatrix} \Delta_{1j}(P) \\ \vdots \\ \Delta_{Jj}(P) \\ \Delta_{\o j}(P) \end{bmatrix} + \big[q_{j1}, \dots, q_{j\bar{a}}\big] \Lambda_j(P) \right) Q_j = \big[q_{j1}, \dots, q_{j\bar{a}}\big] = q_j,$$

where $\Delta_{ij}(P)$, $\Lambda_j(P)$, and q_j are defined in equations (25), (26) and (20) respectively. The matrix Q_j is stochastic, therefore we can effectively get rid of it by right-multiplying both side of the equation with the column of ones $e = [1, \dots, 1]' \in \mathbb{R}^{\bar{a}}$. This is equivalent to taking the sum over all ages between 1 and $\bar{a}$. It also follows from the structure of matrices $\Delta_{ij}(P)$ that the first component in the LHS of the equation above is nothing but the vector of demand for j-type cars from age 1 to $\bar{a}-1$ with the last element equal to the demand for the new cars $D_{j0}(P, q)$ given in equation (21). We then have

$$\Big(\big[D_{j1}(P, q), \dots, D_{j,\bar{a}-1}(P, q), D_{j0}(P, q) \big] + q_j \Lambda_j(P) \Big) e = q_j e.$$

Using the market clearing condition (30) in Theorem 2 we can replace the first $\bar{a}-1$ components in the vector of demands with the corresponding supply. To maintain the matrix notation we express supply given in equation (22) with the help of the keeping choice matrix $\Lambda_i(P)$ defined in equation (26), and the similarly constructed diagonal $\bar{a} \times \bar{a}$ matrix $\Lambda^s_j(P)$ of probabilities to scrap $\Pi(1_s|j, a, P)$. Naturally, $\Lambda^s_j(P)$ has to have 1 in the lower right corner corresponding to the choice probability of scrapping a car of the terminal age $\bar{a}$. The vector of supply for all ages between 1 and $\bar{a}$ is then given by $q_j\big(I - \Lambda_j(P)\big)\big(I - \Lambda_j^s(P)\big)$, with the appropriate zero as the last element. Continuing the derivation we have

$$\Big(q_j - q_j\big(I - \Lambda_j(P)\big)\Lambda_j^s(P) + \Big[0, \dots, 0, D_{j0}(P,q)\Big]\Big)\, e = q_j e.$$

It then immediately follows that

$$q_j\big(I - \Lambda_j(P)\big)\Lambda_j^s(P)e = D_{j0}(P,q),$$

which is the matrix form of the stationary flow condition (32). □

A.4 Proof of Theorem 6 (page 40)

Proof. By the inversion theorem (Lemma 2 Arcidiacono and Miller, 2011) the differences between any two choice specific value functions given in Section 3 are identified. At this point we do not rely on the location normalization of the utilities, but assume that the scale of the extreme value shocks is fixed. There are many ways to show identification of the monetary part of the utility function, but one simple way is the following. All the derivations in the proof can be carried out for each consumer type, we drop the subscript τ for clarity.

Consider the value differences for the choices to purge an existing car given in equation (3) and the choice to remain in the no car state given in equation (8). We have

$$v(i, a, \varnothing, 1_s) - v(\varnothing, \varnothing) = \mu \underline{P}_i, \tag{54}$$

and thus the marginal utility of money μ is point identified (up to a scale).

Next, consider the pairwise differences between the choice specific values $v(i, a, j, d, 1_s)$, $v(i, a, j, d, 0_s)$ and $v(i, a, \kappa)$ given in equation (8). Under the assumption [c] seller transaction costs $T_s(i, a) = 0$, and we have the following linear system of two equations with two unknowns

$$\begin{aligned} v(i, a, j, d, 0_s) - v(i, a, j, d, 1_s) &= -\mu[T_s(i, a) - P_{ia} + \underline{P}_i], \\ v(i, a, i, a, 0_s) - v(i, a, \kappa) &= -\mu[T_s(i, a) + T_b(i, a)]. \end{aligned}$$

Given μ, this system yields identification of the prices and transaction costs for all tradable cars $i \in \{1, \dots, J\}$ and $a \in \{1, \dots, \bar{a} - 1\}$. □

B Solving the Homogeneous Consumer Economy

The limiting case of our model when $\sigma \to 0$ constitutes the *discrete product market* version of Rust (1985a). In this appendix we lay out a simple and efficient numerical solution algorithm for this limiting case, which constitutes the source of precise starting values for the main numerical algorithm in Section 3. We have proven that the following results from Rust (1985a) continue to hold in our discrete setting. Proofs for these results are available on request.

Theorem 1 (Equilibrium in homogeneous consumer economy)**.** *Consider the primary and secondary market for automobiles with one car make/model and homogenous consumers ($\sigma = 0$). Assume infinitely elastic supply of new cars at price $\overline{P}$ and infinitely* *elastic demand for scrapped cars at price $\underline{P}$. The unique stationary equilibrium $\{q, P, a^\star\}$ on this market exists, and is composed of:*

1. *Ownership distribution q, which is the unique invariant distribution corresponding to the physical transition probability matrix Q defined in (27);*
2. *Common scrappage age $a^\star$ equal to the optimal replacement age in the social planner's problem of optimal car replacement (in the absence of secondary market, or equivalently within the class of "buy and hold" strategies);*
3. *Non-increasing price function $P(a)$ defined by*

$$P(a) = \begin{cases} \overline{P} - \frac{1}{\mu}\big(W(0) - W(a)\big), & a \in \{1, \dots, a^\star - 1\}, \\ \underline{P}, & a \geq a^\star, \end{cases} \tag{55}$$

   where $W(a)$ is the unique fixed point of the Bellman operator in the forementioned social planner's replacement problem.

Corollary C1. *In the equilibrium of the homogeneous consumer economy with no transaction costs defined in Theorem 1, consumers are indifferent between replacing their existing car with the car of any age available in the economy. This holds for owners of all ages of cars with positive shares in the stationary fleet age distribution.*

Corollary C2. *The equilibrium in the homogeneous consumer economy with no transaction costs defined in Theorem 1 is welfare maximizing, in particular the discounted expected utility of all consumers is equal to maximum attainable welfare, $V(a) = W(a)$, for all cars with positive shares in the stationary fleet age distribution.*

The main idea of the fast solution algorithm for the homogeneous consumer economy is to express the indifference condition from Corollary C1 as the system of $a^\star - 1$ linear equations to determine the unrestricted prices $P(a)$, $a \in \{1, \dots, a^\star - 1\}$. Because by Corollary C1 consumers are effectively indifferent between any dynamic trading strategies, the strategy of perpetual replacing an existing car of age a results in the maximum attainable expected discounted utility $V(a)$. We have for $a \in \{1, \dots, a^\star - 1\}$

$$V(a) = \tfrac{1}{1-\beta}\Big(u(a) - \beta\mu\big[P(a) - \big(1 - \alpha(a)\big)P(a+1) - \alpha(a)\underline{P}\big]\Big). \tag{56}$$

Let $V(0)$ denote the value of having a new car which is measured right after trading instead of the beginning of the period. Then (56) also holds for $a = 0$, and $V(0) = W(0)$. Thus, using $V(a) - V(a+1) = W(a) - W(a+1)$ (Corollary C2) and the definition of the price function (55), we have for $a \in \{0, \dots, a^\star - 2\}$

$$V(a) - \mu P(a) = V(a+1) - \mu P(a+1), \tag{57}$$

which leads to the following linear equation in prices $(P(a), P(a+1), P(a+2))$:

$$\mu P(a) + \mu\big(\alpha(a)\beta - \beta - 1\big)P(a+1) + \mu\beta\big(1 - \alpha(a+1)\big)P(a+2) = \\ = u(a) - u(a+1) + \beta\mu\underline{P}\big(\alpha(a) - \alpha(a+1)\big). \tag{58}$$

The collection of equations (58) for $a \in \{0, \dots, a^\star - 2\}$ forms the system of $a^\star - 1$ equations with $a^\star - 1$ unknowns, and can be easily solved numerically under our assumption that

$u(a+1) \leq u(a)$ for $a \geq 0$. The system be written in matrix form as

$$X \cdot P = Y, \tag{59}$$

where P is the column vector of prices $\{P(a)\}_{a \in \{1,\ldots,a^\star-1\}}$,

$$X = \begin{bmatrix} \alpha(0)\beta-\beta-1 & \beta-\alpha(1)\beta & 0 & 0 & \cdots & 0 & 0 \\ 1 & \alpha(1)\beta-\beta-1 & \beta-\alpha(2)\beta & 0 & \cdots & 0 & 0 \\ 0 & 1 & \alpha(2)\beta-\beta-1 & \beta-\alpha(3)\beta & \cdots & 0 & 0 \\ 0 & 0 & 1 & \alpha(3)\beta-\beta-1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \alpha(a^\star-3)\beta-\beta-1 & \beta-\alpha(a^\star-2)\beta \\ 0 & 0 & 0 & 0 & \cdots & 1 & \alpha(a^\star-2)\beta-\beta-1 \end{bmatrix},$$

$$Y = \begin{bmatrix} (u(0)-u(1))/\mu + \beta\underline{P}(\alpha(0)-\alpha(1)) - \overline{P} \\ (u(1)-u(2))/\mu + \beta\underline{P}(\alpha(1)-\alpha(2)) \\ (u(2)-u(3))/\mu + \beta\underline{P}(\alpha(2)-\alpha(3)) \\ \vdots \\ (u(a^\star-3)-u(a^\star-2))/\mu + \beta\underline{P}(\alpha(a^\star-3)-\alpha(a^\star-2)) \\ (u(a^\star-2)-u(a^\star-1))/\mu + \beta\underline{P}(\alpha(a^\star-2)-1) \end{bmatrix}.$$

Note that system (59) is well defined for any $a^\star \geq 2$, and consequently a solution to the system denoted $P(a, a^\star)$ can be computed for any value of $a^\star$. We show that the solution of the social planner's problem $W(a)$ corresponds one-to-one to the solution of the linear system (59) under the condition on the price vector $\forall a \in \{1,\ldots,a^\star\} : \underline{P} \leq P(a, a^\star) \leq \overline{P}$ and $P(a^\star, a^\star+1) < \underline{P}$. Because solving a system of linear equations implies much lower computational cost than finding a fixed point $W(a)$, and in particular because the dimensionality of the social planner's problem is larger than that of the linear system, an iterative algorithm that solves (59) for various $a^\star$ to ensure the conditions above hold, is the most efficient numerically.

C Likelihood when accidents are unobserved

Consider the transition probability in a partial information situation where we do not directly observe whether a given car is involved in an accident that leads to its scrappage. As a result we do not fully observe the state x_t representing car holdings of a given household, since for any car state $x_t = (i_t, a_t)$ where $a_t = \bar{a}_{i_t}$ (i.e. cars that have reached the mandatory scrappage state/age $\bar{a}_{i_t}$) we cannot distinguish which of those car are in that state due of an accident (since accidents are not directly observed) and which are in that state not due to an accident, but due to being age $\bar{a}_{i_t} - 1$ in period $t-1$. However we do observe whether the previously chosen car (j_{t-1}, d_{t-1}) was scrapped (exogenously due to accident or endogenously due to voluntary scrappage decision) or not. This implies we fully observe the post-trade ownership state δ_t, and so our approach to inference in the partial information case will be based on the transition probability $P(\delta_{t+1}|\delta_t, \tau, \theta)$.

In the case where $\delta_t = (\emptyset, \zeta_t)$ (either a decision to enter the no car state via a purge decision with an associated scrappage outcome ζ_t, or a decision to remain in the no car state if $x_t = \emptyset$ in which case ζ_t is not relevant), $P(\delta_{t+1}|\delta_t, \tau, \theta)$ is just the conditional choice probability given in equation (13).

If the household did choose to either trade for a car (i_t, a_t) or keep their car (i_t, a_t) at time t, since we do not observe whether that car was involved in an accident between

t and $t+1$, the relevant transition probability $P(\delta_{t+1}|\delta_t,\tau,\theta)$ is a mixture of two choice probabilities, depending on whether the car was involved in an accident or not. Thus, when a scrappage event is observed for the car the household chose at t, then $s_{t+1}=1_s$, (here we let the scrappage indicator, s_{t+1}, equal 1_s if the car the household chose at t was scrapped by period $t+1$, and 0_s otherwise), the transition probability $P(\delta_{t+1}|\delta_t,x_t,\tau,\theta)$ is given by

$$\begin{aligned} P(j_{t+1},d_{t+1},\zeta_{t+1}=1_s|i_t,a_t,\zeta_t,\tau,\theta) &= \Pi(j_{t+1},d_{t+1}|i_t,\bar{a}_{i_t},\tau,\theta)\alpha(i_t,a_t)+ \\ \Pi(j_{t+1},d_{t+1}|i_t,a_t+1,\tau,\theta)&\Pi(s_{t+1}=1_s|j_{t+1},d_{t+1},i_t,a_t+1,\tau,\theta)(1-\alpha(i_t,a_t)), \end{aligned} \quad (60)$$

where $\Pi(j_{t+1},d_{t+1}|i_t,\bar{a}_{i_t},\tau,\theta)$ is the conditional choice probability for a household holding a "clunker" which requires a forced scrappage of the vehicle by our modeling assumptions. Thus, in the event of an accident of the car the houseold chose at time t, we represent it as a transition to the clunker state $\bar{a}$ and in this choice there is no choice over whether or not to sell or scrap the car, so $\Pi(s_{t+1}=1_s|j_{t+1},d_{t+1},i_t,\bar{a},\tau,\theta)=1$. If there is no accident, then the household does have a choice of whether to scrap or sell the car, so in this case we do include the conditional probability of the scrappage choice, $\Pi(s_{t+1}=1_s|j_{t+1},d_{t+1},i_t,a_t+1,\tau,\theta)$ to calculate the overall transition probability in (60) in the event of an observed scrappage of the chosen car between period t and $t+1$.

If there was no scrappage of the car the household chose at time t between periods t and $t+1$, which we denote by $\zeta_{t+1}=0_s$, then $P(\delta_{t+1}|\delta_t,x_t,\tau,\theta)$ is given by

$$\begin{aligned} &P(j_{t+1},d_{t+1},\zeta_{t+1}=0_s|i_t,a_t,\zeta_t,\tau,\theta)= \\ &\quad \Pi(j_{t+1},d_{t+1}|i_t,a_t+1,\tau,\theta)(1-\Pi(s_{t+1}=1_s|j_{t+1},d_{t+1},i_t,a_t+1,\tau,\theta))(1-\alpha(i_t,a_t)), \end{aligned}$$

i.e. it is the probability of choosing to trade the existing car but also choosing not to scrap at $t+1$ conditioning on the event no accident occurred between t and $t+1$ either.

In the event of a choice to keep the current car at time $t+1$, which we have denoted earlier in the paper via the special symbol $(j_{t+1},d_{t+1})=\kappa$ to distinguish it from a decision to trade the current car (i_t,a_t) for another car with the same type and age, i.e. $j_{t+1}=i_t$ and $d_{t+1}=a_t$, we can conclude that no accident and no voluntary scrappage of the previously chosen vehicle could have occurred. Since the keep decision obviates any choice about selling or scrapping the existing car, the transition probability for the keep decision is given by

$$P(j_{t+1}=i_t,d_{t+1}=a_t+1,\zeta_{t+1}|i_t,a_t,\zeta_t,\tau,\theta)=\Pi(\kappa|i_t,a_t+1,\tau,\theta)(1-\alpha(i_t,a_t)). \quad (61)$$

We include the previous car ownership state x_t as a conditioning variable in the transition probability $P(\delta_{t+1}|\delta_t,x_t,\tau,\theta)$ just as we did in the full information case above since when a household decides to keep their current car, we need the information in the incoming car state $x_t=(i_t,a_t)$ to determine the coditional probabilities of a scrappage decision and whether the car will have an accident.

The Kullback-Leibler distance in the case of unobserved accidents is given below

$$\begin{aligned} &D(\theta)\equiv \\ &\sum_\tau\sum_x\sum_\delta\sum_{\delta'}\left[\log\left(P(\delta'|\delta,x,\tau)\right)-\log\left(P(\delta'|\delta,x,\tau,\theta)\right)\right]P(\delta'|\delta,x,\tau)P(\delta|x,\tau)q_\tau(x)f(\tau). \end{aligned} \quad (62)$$

We assume that via direct observation post-trading ownership states, δ_t and δ_{t+1}, it is possible to non-parametrically estimate the transition probability $P(\delta'|\delta, x, \tau)$ for each observed household type τ. Note that though x is not always fully observed, it is observed when a household chooses to keep their car. It is not observed for all decisions involving trading the previous car, however the identity of the new car captured in δ is a sufficient statistic for determining the probability distribution of δ' and in the case of a trade, $P(\delta'|\delta, x, \tau)$ is independent of x given δ. It follows that $P(\delta'|\delta, x, \tau)$ can be non-parametrically estimated in the case where accidents are unobserved and thus the state x is not fully observed.

It is also possible to non-parametrically estimate the cross-sectional distribution of car ownership choices $q_\tau(\delta)$ for each household type τ. This is the stationary distribution of car ownership states, prior to accounting for accidents, whereas $q_\tau(x)$ is the actual incoming stationary distribution of car ownership states, accounting for accidents. It is not hard to show that if $q_\tau(x)$ is the invariant distribution of car ownership states at the start of any period t for household type τ, then $q_\tau(\delta) = \sum_x P(\delta|x, \tau)q_\tau(x)$ is an invariant distribution of household car choices, where $P(\delta|x, \tau)$ is the conditional probability that a household of type τ makes a car ownership decision δ at time $t + 1$ conditional on their car state being x at time t, and where we have added the probability of keeping the current car $x = (i, a)$, denoted by $\delta = (\kappa, \kappa)$, to the conditional probability of trading for a car of type/age $x = (i, a)$. That is, when $\delta = x$ we define $q_\tau(\delta)$ as $\sum_{x'} P(\delta|x', \tau)q_\tau(x') + P(1|x, \tau)q_\tau(x)$. With this redefined δ variable it is no longer necessary to condition on x when writing the conditional choice probability of choosing a car (and scrapping the existing one) at time $t + 1$, we can now simply express it as $P(\delta'|\delta, \tau)$. Then using the invariant distribution over car ownership decisions, $q_\tau(\delta)$, we can write the Kullback-Leibler distance in the case where accidents are not observed as follows

$$D(\theta) \equiv \sum_\tau \sum_\delta \sum_{\delta'} \left[\log\left(P(\delta'|\delta, \tau)\right) - \log\left(P(\delta'|\delta, \tau, \theta)\right)\right] P(\delta'|\delta, \tau)q_\tau(\delta)f(\tau). \tag{63}$$

In summary, in the full information case the relevant transition probability that we use as a basis for estimation is $P(x_{t+1}|x_t, \tau, \theta)$, where x_t is the fully observed car ownership state of the household, expressed as a product of the conditional choice probability for the household's decision over next period car state, and an accident probability that gives the final realized car state x_{t+1} at the start of the next period $t + 1$. When accidents are unobserved, the relevant transition probability is $P(\delta_{t+1}|\delta_t, \tau, \theta)$, using the post-decision state δ_t, which is necessitated by the fact that we do not fully observe the car state x_t at each time period due to the fact that accidents are unobserved.

D Notes on the identification of the model with driving

This appendix is a short note on the identification of the model when we allow for driving, with a linear specification for the predicted optimal driving that can depend on the age of the car. Consider a utility function of the form

$$u(a, x) = \psi_0 + \psi_1 a + \psi_2 a^2 + (\gamma_0 + \gamma_1 a)x - \mu p x + \frac{\phi}{2}x^2 \tag{64}$$

where a is the age of the car, x is vehicle kilometers driven each period, and the parameter vector is $\theta = (\psi_0, \psi_1, \psi_2, \gamma_0, \gamma_1, \mu, \phi)$. We can consider the sum of the first three terms on the right hand side of equation (64) to be the utility of ownership *per se,* i.e. the utility a consumer gets from the pure ownership (and option value to drive) even if the household does not do any driving, $x = 0$. The remaining components are the net utility from driving, after deducting the cost of driving (translated into utility terms by multiplying by the marginal utility of money, μ). There are seven parameters to be identified for each consumer type/car type in the model, and for notational simplicity we have suppressed the dependence of all θ parameters except for μ (the marginal utility of money or price-responsiveness coefficient) since we assume that μ depends on consumer type τ but not on the car type j.

We want to consider the identification of these parameters from observations on: 1) consumer trading of automobiles, and 2) observed driving. The symbol p is the per kilometer cost of fuel plus any taxes, and in our model there is *no variation in this price, either over time or across consumers.* However there is variation in p across car types due to different fuel efficiency of different types and ages of cars. However we really cannot use variation in p as a source of identification of the model parameters if we are being fully faithful to the model, which currently does not allow any time series or cross sectional variation in p except over car types as noted above. Fortunately, we now show that we can identify the parameters using the information on car trading, and in a way that is "just identified" so we don't face a trade-off in terms of fitting the model of car driving or the model of car trading: each can be estimated separately and the structural parameters that imply the best-fitting "reduced-form" specifications for driving and car trading are derived below.

We derive the indirect utility function first, by using the utility function above to calculate the optimal level of driving,

$$x^*(p, a) = -\frac{1}{\phi}\left[\gamma_0 + \gamma_1 a - \mu p\right]. \tag{65}$$

Equation (65) is the "structural driving equation" implied by the direct utility function $u(x, a)$ in equation (64). But there is a corresponding "reduced form" or unrestricted driving equation which we denote by

$$x = d_0 + d_1 a + d_2 p. \tag{66}$$

Though there is no identifying variation in p we do have identifying variation in a, so we can separately identify the constant term in the driving equation in (65) and the age coefficient. So we can treat the age coefficient γ_1/ϕ as known given knowledge of observed driving $x^*(p, a)$ and also the constant term $[\gamma_0 - \mu p]/\phi$, though at this point we cannot separately identify all the parameters from only these two identified coefficients from the "driving equation." Note that optimal driving must be positive, so this implies an additional inequality restriction on the parameters,

$$\gamma_0 + \gamma_1 a \geq \mu p, \quad a = \{0, 1, \dots, \bar{a} - 1\} \tag{67}$$

and if, as we expect, $\gamma_1 < 0$, then the set of restrictions below can be reduced to this single inequality restriction at the last age consumers are allowed to own cars

$$\gamma_0 + \gamma_1(\bar{a} - 1) \geq \mu p \tag{68}$$

but if this inequality is violated, then it is easy to see that $x^*(p,a) = 0$ and the consumer gets zero utility from owning a car (and hence absent extreme value shocks would not buy one since the utility from the outside good is normalized to zero, and there are purchase price and transactions costs involved in buying a car).

Now plugging the optimal driving back into utility, we can derive the indirect utility function,

$$u(a, x^*(p,a)) = v(a,p,\tau) = \psi_0 + \psi_1 a + \psi_2 a^2 - \frac{1}{2\phi}\left[\gamma_0 + \gamma_1 a - \mu p\right]^2 \tag{69}$$

$$= u_0 + u_1 a + u_2 a^2 \tag{70}$$

where

$$u_0 = \psi_0 - \frac{1}{2\phi}\left[\gamma_0 - \mu p\right]^2, \qquad u_1 = \psi_1 - \frac{\gamma_1}{\phi}\left[\gamma_0 - \mu p\right], \qquad u_2 = \psi_2 - \frac{1}{2\phi}\left[\gamma_1^2\right].$$

We can consider the coefficients (u_0, u_1, u_2) and the marginal utility of money as identified from an unrestricted or "reduced form" dynamic discrete choice model of car trading. We now argue that the seven parameter specification of utility $u(x,a,\theta)$ given in equation (64) is just identified from unrestricted estimation of the reduced-form driving equation (66) and the dynamic discrete choice model of car trading. First, assume we can identify the marginal utility of money, μ, from estimation of the latter model. Then there are only 6 remaining structural parameters to be identified, $(\psi_0, \psi_1, \psi_2, \gamma_0, \gamma_1, \phi)$ and these are determined from the following 6 equations that provide enough flexibility to ensure perfect unrestricted fit of both the reduced-form driving equation (parameters (d_0, d_1, d_2)) and the reduced-form dynamic discrete choice model (parameters (u_0, u_1, u_2) plus the marginal utility parameter μ).

$$d_0 = -\frac{\gamma_0}{\phi}, \qquad d_1 = -\frac{\gamma_1}{\phi}, \qquad d_2 = \frac{\mu}{\phi}$$

Thus, given the estimate of the marginal utility of money $\hat{\mu}$ from the dynamic discrete choice model, we can back out $\hat{\phi}$ from the last equation of (71), and thus also $(\hat{\gamma}_0, \hat{\gamma}_1)$. Then given these parameter estimates we can determine the parameters $(\hat{\psi}_0, \hat{\psi}_1, \hat{\psi}_2)$ from equation (71) in a way that entails no restrictions on the estimation of the coefficients (d_0, d_1, d_2) of the reduced-form driving model (66) or the coefficients (μ, u_0, u_1, u_2) of the reduced-form dynamic discrete choice model of car trading.

To derive (70) we assumed that inequality (68) holds so that the consumer would want to do some positive driving at all possible ages. However if the inequality does not hold at all ages, then $u(a, x^*(p,a))$ is given by (70) only for ages a satisfying inequality (67) and for all higher ages $a \geq \hat{a}$ (where $\hat{a}$ is the largest age for which inequality (70) holds), then $x^*(x,p) = u(a, x^*(p,a)) = 0$ for all $a > \hat{a}$. Notice that $u_0 \geq 0$ and $u_2 \geq 0$ and if $\gamma_1 < 0$, then $u_1 \leq 0$. Thus, consumer preferences over cars of different ages are expected to be decreasing and convex in the age of the car. The strict convexity in age is required in order to imply a finite amount of driving: i.e. if $\phi = 0$ (so indirect utility is linear in a) then $x^*(p,a)$ is predicted to be infinite for any a where the inequality restriction (67) is strict.

Now let's suppose that we could identify the four parameters (u_0, u_1, u_2, μ) by estimating a model of automobile trading that ignored driving, but with a utility function that is quadratic in the age of the car as in equation (70). We presume that the marginal utility of money parameter μ is identified by the variation in trading cars of different ages using known prices of used cars, $P(a)$, $a = 0, \ldots, P(\bar{a} - 1)$ that are treated as data. Even though these prices are the same for all consumers, there is variation over ages of different cars and combined with the scrappage decision, we will assume these coefficients can be identified.

Now given the additional observation on driving are the other coefficients $(\gamma_0, \gamma_1, \phi)$ identified? The answer is yes. As we noted above, the ratio γ_1/ϕ is identified from the driving equation (65), and the coefficient γ_1^2/ϕ is identified as the quadratic coefficient u_2 from the dynamic discrete choice model of vehicle trading with the quadratic in age utility function given in (70). Thus γ_1 is identified as the ratio of these two coefficient estimates. Next, given γ_1 we can identify ϕ from either the linear term of the driving equation, $-\gamma_1/\phi$ or the coefficient u_2 on the a^2 term of the utility of the dynamic discrete choice model, $u_2 = -\gamma_1^2/(2\phi)$. Then using the equation for u_1, the coefficient on a in the dynamic discrete choice model, we can identify γ_0. Thus the parameter vector $\theta = (\gamma_0, \gamma_1, \mu, \phi)$ is in fact overidentified since there are additional restrictions implied by the parameters via the constant term in the driving equation, $x^*(0, p)$ and the constant term u_0 in the utility function in the dynamic discrete choice model.

These parameters could be estimated by jointly by maximum likelihood, if we treat observed driving as affected by measurement error. But doing this would require estimation of an additional parameter for the variance of the measurement error in driving. We would also have to take seriously the inequality restriction (67) and make sure that we don't ignore it and get strange results from squaring negative predicted driving, rather than carefully obeying the inequality restriction that implies cars provide zero direct utility when predicted driving is negative.

To get started, I would favor an "indirect least squares" sort of approach where we just estimate the coefficients (u_0, u_1, u_2, μ) of the dynamic discrete choice ingoring driving and make sure we can fit the pattern of trading in cars well. Then with these estimated, we can estimate a linear driving equation and "back out" the implied $(\gamma_0, \gamma_1, \phi)$ coefficients via an "indirect least squares" approach and check that they are reasonable. This would be the "second step". Finally for more efficient parameter estimates, we could estimate the model "structurally" (either by ML or by MM) that impose the "cross equation restrictions" from the observations on driving and observations on car trading using the indirect least squares parameters as starting points and then recasting the parameters of the model directly in terms of the "deep structural parameters" that allow for driving, $\theta = (\gamma_0, \gamma_1, \mu, \phi)$.

E Estimation details

E.1 Data and Institutional Details

The data comes from the Danish demographic registers and covers the period 1996 to 2008. The dataset covers the universe of all Danish households and all cars owned by private individuals. Driving information is obtained from odometer readings that occur when cars are taken to mandatory driving inspections biannually starting from a car age of four.[46] Fuel prices come from eof.dk and are a country-level average.

[46] That is, a car is inspected at ages 4, 6, 8, 10, and so forth.

Car types: Cars are aggregated into four discrete types: we first split cars based on whether the car's weight is above or below the median for that car's vintage cohort, and then within each of those two subsamples, we further split cars based on whether they are above or below the median fuel efficiency. This way, we get four classes of cars, that we name "green" or "brown" for high and low fuel efficiency respectively, and "heavy" and "light" for high and low weights, respectively. Making the splits separately by vintage has the benefit that the distribution of car types is roughly constant over time, but has the drawback that car attributes for the four classes are not constant.[47] Therefore, we simply take the average of the car attributes for each of the four types in our model. Finally, we split cars into 25 age groups from brand new to 24 years old, where the last age category captures cars of 24 years or older. Table 3 presents key summary statistics for our four car types, aggregated over all the years of our sample as well as all the different car ages.

Household types: Households are split into 8 different types based on whether the household is a couple or single, has high or low work distance, and whether income is high or low based. Income splits are based on the median income in the demographic cell. Work distance comes from the Danish tax deduction for travel distance to work. This deduction is only applicable for full-time workers living further than 12 km from their work place (each way), and slightly under half of Danes have high work distance by this definition, although it differs quite a bit by cohabitation status. Table 2 presents summary statistics, where we have computed the weighted average over the years of our data for each household type to simplify the exposition. We also present averages for the number of kids in the household.

Prices: Recall that our model takes the new car price and the scrap price as given. Towards this end, we leverage data on the MSRPs to construct new car prices, which we take as the weighted averages of all underlying car types matched to each of our four discretized types, j. We construct an estimate of the scrappage price by leveraging data on suggested annual depreciation rates of 87% that we have from the Danish Automobile Dealer Association.[48] We construct these numbers for all years 1998 to 2008 and take the unweighted average over years.

The registration tax paid upon the purchase of a new car in Denmark is among the highest in the world. It is a linear tax that has a kink, K, with one rate, τ_1, below K and a higher rate, $\tau_2 > \tau_1$, applying to any price above K. Finally, 25% VAT is paid of the price including the tax. So if the price before the registration fee and VAT is given by P, then the registration fee to be paid is given by $T(P) = \tau_1 \min(1.25P, K) + \tau_2 \max(1.25P - K, 0)$. In 2008, K was 81,000 DKK (around 16,000 USD), τ_1 was 105% and τ_2 was 180%. In our counterfactuals in Table 1, we lower τ_1 and τ_2 to half their initial values, i.e. 52.5% and 90% respectively. There are also annual taxes for car ownership as well as mandatory insurance costs, which we abstract from.

We use a social cost of carbon of US$50/ton (290 DKK) and the other external costs per kilometer travelled are valued at 0.6216 DKK/km and they consist of noise, accidents, congestion and local air pollution, as measured by Transport (2010).

We do not observe scrappage in our data *per se.* Instead, we define a vehicle as having been scrapped if an ownership spell ends and no other ownership spell ever begins afterwards. Since our extract of the ownership register comes from September of 2011

[47]In reality, technological progress implies that car attributes are improving over time as engines can drive further per liter of fuel. To accomodate our model's stationary nature, we ignore this "attribute inflation".

[48]The rates vary by car type but that variation is negligible, especially compared to the variation in new car prices across car types.

Table 2: Summary Statistics for Households

τ	Name	N	Income	1(Single)	Work distance	Age	1(Urban)	No. kids
1	Low WD, Couple, Poor	6,500,464	311.68	0.00	0.00	55.03	0.22	0.48
2	Low WD, Couple, Rich	6,352,821	777.19	0.00	0.00	46.38	0.21	1.03
3	Low WD, Single, Poor	7,906,100	109.92	1.00	0.00	54.21	0.35	0.11
4	Low WD, Single, Rich	7,666,452	301.15	1.00	0.00	48.21	0.33	0.20
5	High WD, Couple, Poor	4,031,412	494.61	0.00	34.63	40.58	0.12	0.99
6	High WD, Couple, Rich	3,862,441	862.43	0.00	42.13	43.57	0.12	1.21
7	High WD, Single, Poor	1,217,611	215.04	1.00	26.71	33.85	0.25	0.22
8	High WD, Single, Rich	1,171,919	413.24	1.00	32.98	41.14	0.22	0.24

Note: The column "N" denotes the observations of each household type available across all the years, 1996–2008. The remaining variables are all weighted averages of the corresponding variables with the annual observation counts as weights. Household types are defined based on splitting the sample into cells based on single/couple status, whether work distance is zero or positive, and finally splitting households in two depending on income within the cell is above or below the median. Work distance is based on a travel tax deduction, and it is only positive if one of the household members has more than 12 km to work (each way), and so it is naturally zero for unemployed. The urban dummy is equal to one for the six largest cities in Denmark: Copenhagen, Frederiksberg, Aarhus, Aalborg, and Odense.

Table 3: Summary Statistics for Cars

	No car	light, brown	light, green	heavy, brown	heavy, green
Obs.	16895290	4683737	5594897	5351904	6183392
Diesel share		0.00	0.08	0.14	0.21
Depreciation Factor		0.87	0.87	0.87	0.87
Weight (tons)		1.42	1.28	1.96	1.64
Variables used in the model					
Price, new (1000 DKK)		174.90	144.55	299.45	253.40
Price, new excl. tax (1000 DKK)		67.33	56.41	102.91	89.76
Price, scrap (1000 DKK)		6.20	5.26	9.35	8.76
Fuel efficiency (km/l)		12.84	15.06	9.89	12.63

Note: The four car categories are defined by first splitting cars into two groups based on weight, and then on fuel efficiency within each weight sub-group. The splits are made separately for every car vintage, implying that the attributes of, say, a "light, green" car is changing over time. The variable "Depreciation Factor" is a suggested annual depreciation factor set by the Danish Automobile Dealer Association. The rate varies across cars but not over time, implying that the association uses a constant exponential discounting rule.

Table 4: First-stage OLS Estimates of the Driving Model

Dependent variable: thousands of kilometers driven per year			
γ_0	Intercept	19.07	(0.58)
$\hat{\gamma}_1^a/\phi_\tau$	Car age	-0.1325	(0.03)
$\hat{\gamma}_2^a/\phi_\tau$	Car age squared	-0.001975	(0.00)
$\hat{\gamma}_\tau/\phi_\tau$	Intercept, Low WD, Couple, Rich	4.43	(0.70)
$\hat{\gamma}_\tau/\phi_\tau$	Intercept, Low WD, Single, Poor	-3.752	(1.24)
$\hat{\gamma}_\tau/\phi_\tau$	Intercept, Low WD, Single, Rich	-0.0325	(0.79)
$\hat{\gamma}_\tau/\phi_\tau$	Intercept, High WD, Couple, Poor	9.825	(0.79)
$\hat{\gamma}_\tau/\phi_\tau$	Intercept, High WD, Couple, Rich	12.33	(0.74)
$\hat{\gamma}_\tau/\phi_\tau$	Intercept, High WD, Single, Poor	6.436	(1.52)
$\hat{\gamma}_\tau/\phi_\tau$	Intercept, High WD, Single, Rich	12.23	(1.27)
$\hat{\gamma}_j/\phi_\tau$	Car dummy: light, green	-1.994	(0.15)
$\hat{\gamma}_j/\phi_\tau$	Car dummy: heavy, brown	4.345	(0.15)
$\hat{\gamma}_j/\phi_\tau$	Car dummy: heavy, green	3.606	(0.14)
μ/ϕ	Price (common)	-7.074	(0.84)
$\hat{\mu}_\tau/\phi_\tau$	Price, Low WD, Couple, Rich	-4.111	(1.02)
$\hat{\mu}_\tau/\phi_\tau$	Price, Low WD, Single, Poor	4.732	(1.84)
$\hat{\mu}_\tau/\phi_\tau$	Price, Low WD, Single, Rich	0.2781	(1.16)
$\hat{\mu}_\tau/\phi_\tau$	Price, High WD, Couple, Poor	-6.41	(1.17)
$\hat{\mu}_\tau/\phi_\tau$	Price, High WD, Couple, Rich	-9.892	(1.09)
$\hat{\mu}_\tau/\phi_\tau$	Price, High WD, Single, Poor	-1.714	(2.29)
$\hat{\mu}_\tau/\phi_\tau$	Price, High WD, Single, Rich	-9.007	(1.91)
N	Driving periods	19,635,940	

while our last sample year is 2008, this means that a car should have been without owner for 3 years, which typically means it has been scrapped. Note also that we do not observe whether a car was involved in an accident in the data, although our model will make a distinction between accidents and voluntary scrappage decisions.

E.2 Estimation

As explained in Section 6, estimation is composed of three steps:

1. Estimate the reduced form driving parameters from (66) using linear regression: the estimates are in Table 4.
2. Estimate the reduced form dynamic discrete choice parameters from (70) using Maximum Likelihood: the estimates are in Tables 4, 5 and 7 to 10.
3. Back out the "deep structural parameters" $\theta_{\tau,j} = (\psi_{\tau,j,0}, \psi_{\tau,j,1}, \psi_{\tau,j,2}, \gamma_{\tau,j,0}, \gamma_{\tau,j,1}, \phi_{\tau,j})$ for each of the 8 consumer types τ and 4 car types j using equations (71) and (71) of Appendix D.

Table 5: Estimates: Accidents

	light, brown	light, green	heavy, brown	heavy, green
Intercept	-5.5876	-6.0006	-5.6697	-5.7399
	(0.0122)	(0.0109)	(0.0096)	(0.0090)
Age slope	0.1725	0.2134	0.2007	0.1969
	(0.0016)	(0.0013)	(0.0009)	(0.0009)

Table 6: Estimates: Scrappage Decision

	Estimate
σ_s: Scrap utility error variance	0.3852
	(0.3852)
Intercept: selling (baseline is scrapping)	-1.5999
	(-1.5999)
Selling in inspection years	-2.2955
	(-2.2955)

Table 7: Estimates: Marginal Utility of Money

	μ_τ: marginal utility of money
Low WD, Couple, Poor	0.1074
	(0.0006)
Low WD, Couple, Rich	0.1059
	(0.0006)
Low WD, Single, Poor	0.0895
	(0.0007)
Low WD, Single, Rich	0.1024
	(0.0006)
High WD, Couple, Poor	0.0980
	(0.0006)
High WD, Couple, Rich	0.1091
	(0.0006)
High WD, Single, Poor	0.0890
	(0.0007)
High WD, Single, Rich	0.1028
	(0.0007)

Table 8: Estimates: Utility Intercept

$u_{\tau,j,0}$: intercept in indirect utility for car ownership				
	light, brown	light, green	heavy, brown	heavy, green
Low WD, Couple, Poor	3.5100	2.9687	4.8387	4.5461
	(0.0162)	(0.0152)	(0.0253)	(0.0231)
Low WD, Couple, Rich	3.8870	3.3155	5.4279	5.1089
	(0.0155)	(0.0146)	(0.0242)	(0.0220)
Low WD, Single, Poor	2.3158	2.0491	3.3213	3.0367
	(0.0177)	(0.0157)	(0.0272)	(0.0251)
Low WD, Single, Rich	3.1192	2.6901	4.4031	4.0691
	(0.0160)	(0.0148)	(0.0249)	(0.0227)
High WD, Couple, Poor	3.7463	3.2959	5.0193	4.8326
	(0.0149)	(0.0138)	(0.0231)	(0.0210)
High WD, Couple, Rich	4.6059	4.1597	6.2197	5.9814
	(0.0165)	(0.0154)	(0.0255)	(0.0232)
High WD, Single, Poor	2.6024	2.3494	3.5814	3.3905
	(0.0195)	(0.0169)	(0.0302)	(0.0276)
High WD, Single, Rich	3.4274	3.0413	4.7046	4.4860
	(0.0189)	(0.0169)	(0.0290)	(0.0265)

Table 9: Estimates: Utility Age Slope

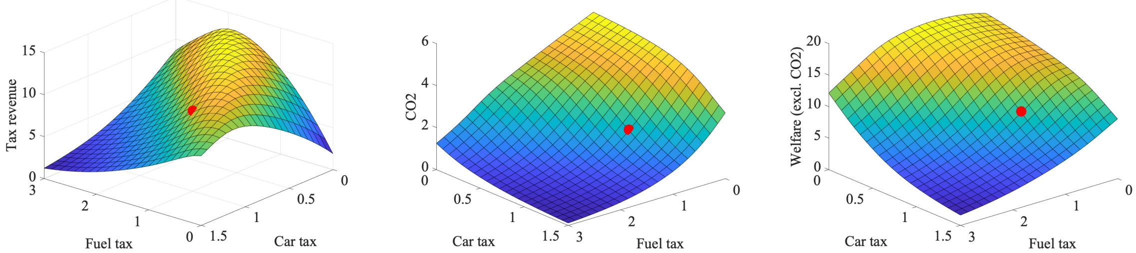
$u_{\tau,j,1}$: coefficient on age in indirect utility for car ownership				
	light, brown	light, green	heavy, brown	heavy, green
Low WD, Couple, Poor	-0.1397	-0.0914	-0.2084	-0.1651
	(0.0008)	(0.0009)	(0.0013)	(0.0011)
Low WD, Couple, Rich	-0.1520	-0.0973	-0.2294	-0.1881
	(0.0008)	(0.0009)	(0.0012)	(0.0010)
Low WD, Single, Poor	-0.0929	-0.0607	-0.1507	-0.1076
	(0.0009)	(0.0009)	(0.0013)	(0.0011)
Low WD, Single, Rich	-0.1253	-0.0836	-0.1955	-0.1487
	(0.0008)	(0.0009)	(0.0012)	(0.0010)
High WD, Couple, Poor	-0.1329	-0.0814	-0.2023	-0.1629
	(0.0008)	(0.0009)	(0.0011)	(0.0010)
High WD, Couple, Rich	-0.1570	-0.1055	-0.2423	-0.1995
	(0.0008)	(0.0009)	(0.0013)	(0.0011)
High WD, Single, Poor	-0.1082	-0.0718	-0.1675	-0.1249
	(0.0010)	(0.0009)	(0.0015)	(0.0012)
High WD, Single, Rich	-0.1436	-0.1001	-0.2105	-0.1673
	(0.0010)	(0.0010)	(0.0015)	(0.0012)

Table 10: Estimates: Transaction Costs (in utility terms)

Utility cost of transacting		
	Intercept	No car
Low WD, Couple, Poor	6.8509	1.7893
	(0.0221)	(0.0029)
Low WD, Couple, Rich	6.6965	1.0742
	(0.0221)	(0.0030)
Low WD, Single, Poor	6.7670	3.0771
	(0.0185)	(0.0045)
Low WD, Single, Rich	6.8514	2.5729
	(0.0209)	(0.0031)
High WD, Couple, Poor	6.4970	0.7784
	(0.0203)	(0.0036)
High WD, Couple, Rich	6.6843	0.1720
	(0.0227)	(0.0045)
High WD, Single, Poor	6.2479	2.3367
	(0.0192)	(0.0066)
High WD, Single, Rich	6.5250	1.7912
	(0.0216)	(0.0064)

Figure 9: The Effects of Varying the Fuel and Registration Tax Rates

(a) Total tax revenue (b) CO_2 Emissions (c) Social Surplus



Note: All three panels have the same x and y axes, namely the tax rate for fuel and car registrations respectively, normalized by the sample values so that the baseline outcomes occur at (1,1). The panels differ in terms of the rotation and which outcome is on the z axis: tax revenue, CO_2 emissions, and social surplus respectively. Social surplus is the sum of consumer surplus and tax revenue, subtracting the external costs of driving (accidents, congestion, etc.) including CO_2 valued at \$50/ton.